# GRAPHS OF LATTICES IN REPRESENTATIONS OF FINITE GROUPS

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# GRAPHS OF LATTICES IN REPRESENTATIONS OF FINITE GROUPS

By

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The undersigned hereby certify that they have read and recommend to the Faculty of Graduate Studies for acceptance a thesis entitled "**Graphs of lattices in representations of finite groups**" by **Marica Knezevic** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy**.

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*To Lambros, the love of my life*

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# Abstract

This thesis work is motivated by the Langlands program, which relates objects from number theory and representation theory. In particular it is motivated by the compatibility of the local and global cases, especially in the mod $p$ case. Given a finite group $G$, $E$ a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_E$, uniformizer $\pi_E$, $V$ a finite dimensional $E$-vector space and $\rho : G \to \mathrm{Aut}_E(V)$ an absolutely irreducible representation, we associate to $\rho$ the following directed graph: its vertices are homothety classes of lattices in $V$ and there is an edge from (the class of) $\Lambda$ to $\Lambda'$ when $\pi_E\Lambda \subseteq \Lambda' \subseteq \Lambda$ and the quotient $\Lambda/\Lambda'$ is irreducible. We also label the edge according to the isomorphism class of $\Lambda/\Lambda'$.

The central theme of this thesis is the study of stable lattices in $p$-adic representations and their corresponding graphs. In particular, we show certain properties of these associated graphs, including finiteness, connectedness, a duality property and that the length of a cycle is a multiple of the number of the Jordan-Holder factors. Moreover, we restrict our attention to certain families of representations arising from admissible representations of $\mathrm{GL}_2$ of a $p$-adic field and show further properties of their graphs. In the case where $\rho$ is of principal series type we compute the bound, that is we find an explicit integer $c$ and the lattice $\Lambda$ in $V$ such that all lattices $\Lambda'$ in $V$ up to homothety satisfy $\pi_E^c\Lambda \subseteq \Lambda' \subseteq \Lambda$. In the case where $\rho$ is of tame principal series type we compute the graphs and investigate their properties. We also compute graphs for certain representations of interest, where most of them are obtained using Magma. The Magma code is attached in the thesis.

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# Chapter 1

# Introduction & Preliminaries

In this section we comment on the motivation for the thesis (which is not logically necessary for the results in the thesis). Let $L$ be a totally real field in which $p$ is unramified and $\rho : \mathrm{Gal}(\bar{\mathbb{Q}}/L) \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ a continous, totally odd and irreducible representation. We can associate to $\rho$ a $\bar{\mathbb{F}}_p$ representation of a completed totally definite quaternion algebra over $L$. We follow the definition of that associated representation as it is done in [3] and [6]. Let $Q$ be a totally definite quaternion algebra over $L$, that is for every archimedian place $\ell$ of $L$, $Q_\ell$ is the Hamilton division algebra. Also assume that $Q$ has centre $L$ and is split at all $v|p$. denote by $\mathbb{A}_L^f$ the finite adeles of $L$. Let $\psi = \Pi_v \psi_v : L^\times \backslash (\mathbb{A}_L^f)^\times \to \bar{\mathbb{F}}_p$ be such that $det(\rho) = \psi\omega$ via the reciprocity map normalised such that $p$ goes to the geometric Frobenius, where $\omega$ is the reduction modulo $p$ of the $p$-adic cyclotomic character. We can associate to $\rho$ a representation of $(Q \otimes_L \mathbb{A}_L^f)^\times$ in the following way: for $U \subset (Q \otimes_L \mathbb{A}_L^f)^\times$ open compact subgroup such that $\psi \mid_{U \cap (\mathbb{A}_L^f)^\times} = 1$ we denote by $S_\psi^Q(U, \bar{\mathbb{F}}_p)$ the space of functions $f : Q^\times \backslash (Q \otimes_L \mathbb{A}_L^f)^\times / U \to \bar{\mathbb{F}}_p$ such that $f(aq) = \psi(a) f(q)$ for any $a \in (\mathbb{A}_L^f)^\times$, $q \in (Q \otimes_L \mathbb{A}_L^f)^\times$. For open compact subgroups $U \subset V \subset (Q \otimes_L \mathbb{A}_L^f)^\times$ we have natural maps $S_\psi^Q(V, \bar{\mathbb{F}}_p) \to S_\psi^Q(U, \bar{\mathbb{F}}_p)$ induced by projection. Define $S_\psi^Q(\bar{\mathbb{F}}_p) := \varinjlim_{U} S_\psi^Q(U, \bar{\mathbb{F}}_p)$.

We have a left action of $(Q \otimes_L \mathbb{A}_L^f)^\times$ on $S_\psi^Q(\bar{\mathbb{F}}_p)$ given by translating functions on the right: for any $f \in S_\psi^Q(\bar{\mathbb{F}}_p)$ and any $g \in (Q \otimes_L \mathbb{A}_L^f)^\times$, $gf(x) := f(xg)$ which is well defined. For $g \in (Q \otimes_L \mathbb{A}_L^f)^\times$, $U, V \subset (Q \otimes_L \mathbb{A}_L^f)^\times$ open and compact we have double coset operators $[VgU] : S_\psi^Q(U, \bar{\mathbb{F}}_p) \to S_\psi^Q(V, \bar{\mathbb{F}}_p)$ defined in the following way. For a set of representatives $g_i \in (Q \otimes_L \mathbb{A}_L^f)^\times$ such that $VgU = \coprod_i g_i U$, we define $[VgU](f)(x) := \sum_i (g_i f)(x)$, $x \in (Q \otimes_L \mathbb{A}_L^f)^\times$. It is easy to see that the definition of $[VgU]$ does not depend on the choice of $g_i$'s. For any $v \in V$ we have that $VgU = vVgU = \bigcup_i vg_i U$ so $vg_i$'s are also a set of representatives which implies that $[VgU](f)$ is $V$ invariant, that is $[VgU](f) \in S_\psi^Q(V, \bar{\mathbb{F}}_p)$. For any prime $\ell$ such that $Q$ is split at $\ell$ and $\mathrm{GL}_2(\mathcal{O}_{L,\ell}) \subset U$ let us define $T_\ell^U := [U \begin{pmatrix} \varpi_\ell & 0 \\ 0 & 1 \end{pmatrix} U]$. For open and compact $U \subset (Q \otimes_L \mathbb{A}_L^f)^\times$, let $R_U$ be a finite set containing all the primes $\ell$ such that $Q$ is ramified at $\ell$, $\mathrm{GL}_2(\mathcal{O}_{L,\ell}) \nsubseteq U$, $\rho$ is ramified at $\ell$ or $\ell$ divides $p$. denote by $T_\psi^{R_U}(U)$ the commutative $\bar{\mathbb{F}}_p$ subalgebra of $\mathrm{End}_{\bar{\mathbb{F}}_p}(S_\psi^Q(U, \bar{\mathbb{F}}_p))$ generated by $T_\ell^U$ for all $\ell \notin R_U$. denote by $m_\rho^{R_U}$ the ideal in $T_\psi^{R_U}(U)$ generated by $T_\ell^U - \psi(Fr_\ell)^{-1} tr(\rho_\ell(Frob_\ell))$ where $Fr_\ell$ is the arithmetic Frobenius at $\ell$ for all $\ell \notin R_U$. Define $S_\psi^Q(U, \bar{\mathbb{F}}_p)[m_\rho^{R_U}]$ to be the set of all functions $f \in S_\psi^Q(U, \bar{\mathbb{F}}_p)$ such that $Tf = 0$ for any $T \in m_\rho^{R_U}$. It is shown in [3] that $S_\psi^Q(U, \bar{\mathbb{F}}_p)[m_\rho^{R_U}]$ does not depend on the choice of $R_U$, so we can write it as $S_\psi^Q(U, \bar{\mathbb{F}}_p)[m_\rho]$. For $U, V$ compact open subgroups of $(Q \otimes_L \mathbb{A}_L^f)^\times$ such that $V \subset gUg^{-1}$ we have that $VgU \subset gU$ which implies that $VgU = gU$. This implies that in the case when $V \subset gUg^{-1}$ we have that $[VgU] : S_\psi^Q(U, \bar{\mathbb{F}}_p) \to S_\psi^Q(V, \bar{\mathbb{F}}_p)$ is just a right translation by $g$. Take $R_V$ big enough, such that $R_U = R_V$ and that $g_\ell \in \mathrm{GL}_2(\mathcal{O}_\ell) \subset V$ for any $\ell \notin R_V$. We get that $[VgU]$ sends $S_\psi^Q(U, \bar{\mathbb{F}}_p)[m_\rho]$ to $S_\psi^Q(V, \bar{\mathbb{F}}_p)[m_\rho]$. So now we can define $S_\psi^Q(\bar{\mathbb{F}}_p)[m_\rho] := \lim\limits_{\overrightarrow{U}} S_\psi^Q(U, \bar{\mathbb{F}}_p)[m_\rho]$.

An important part of the Langlands programme is to understand the relationship between $\rho$ and $S_\psi^Q(\bar{\mathbb{F}}_p)[m_\rho]$ in terms of their local behavior. In [3] it is conjectured that $S_\psi^Q(\bar{\mathbb{F}}_p)[m_\rho]$ is isomorphic to a restricted tensor product $\otimes'_\ell \pi_\ell$ where $\pi_\ell$ is a smooth admissible representation of $(Q \otimes_L L_\ell)^\times$ over $\bar{\mathbb{F}}_p$. Especially in the case when $\ell$ divides $p$, $\pi_\ell$ is a $(Q \otimes_L L_\ell)^\times \cong \mathrm{GL}_2(L_\ell)$ representation over $\bar{\mathbb{F}}_p$ with $\mathrm{GL}_2(\mathcal{O}_{L_\ell})$ socle $\oplus_{\sigma \in D(\rho_\ell)} \sigma$ ($D(\rho_\ell)$ is the set of Diamond weights associated to $\rho_\ell$ as in [7]). Let us fix $\ell \mid p$ and assume that $\rho_\ell$ is generic. In [6] the author shows that we can embed in $\pi_\ell$ a representation $D_0(\rho_\ell)$ of $\mathrm{GL}_2(\mathcal{O}_\ell)$ over $\bar{\mathbb{F}}_p$ ($D_0(\rho_\ell)$ as in [7] proposition 13.1). Breuil also conjectured in [6] that $D_0(\rho_\ell)$ as a $\mathrm{GL}_2(\mathcal{O}_{L_\ell})$ subrepresentation of $\pi_\ell$ is such that its $I_{1,\ell}$ invariant subspace is stable under the action of $\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} \in \mathrm{GL}_2(L_\ell)$ given from $\pi_\ell$ ($I_{1,\ell}$ is the subgroup of $\mathrm{GL}_2(\mathcal{O}_{L_\ell})$ consisting of upper unipotent matrices modulo $p$). This is proved by Emerton, Gee and Savitt in [14] under the assumption that the conjecture in [3] is true. Let us define a Diamond diagram as in [7], theorem 13.8. So we have that the Diamond diagram $(D_0(\rho_\ell), D_1(\rho_\ell), can)$ is embedded in the diagram $(\pi_\ell, (\pi_\ell)^{I_{1,\ell}}, can)$ (where $D_1(\rho_\ell) := (D_0(\rho_\ell))^{I_{1,\ell}}$ with an action of $\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} \in \mathrm{GL}_2(L_\ell)$ given from $\pi_\ell$ and $can$ the canonical embedding). Breuil in [4] (definition 4.3) defines an operator $S^{d_i} : V_{\chi_i} \to V_{\chi_i}$, where $V_\chi \subset (socD_0)^{I_1}$ is the isotypic subspace associated to the character $\chi_i$. Because $V_{\chi_i}$ is one dimensional, the map $S^{d_i}$ is multiplication with some constant. Denote this constant with $\nu_\sigma$, where $\chi = \sigma^{I_1}$. In [4] Breuil conjectures that when $\rho_\ell$ is irreducible

$$\nu_\sigma = (-1)^{\frac{d_\sigma}{2} + \frac{d_\sigma h_\sigma (1 + \sum_{j=0}^{f-1} r_j)}{2f}}$$

and in the case when $\rho_\ell$ is reducible

$$\nu_\sigma = (-1)^{\frac{d_\sigma h_\sigma}{f} \sum_{j=0}^{f-1} r_j} \alpha^{(|\bar{J}_\sigma| - |J_\sigma|)\frac{d_\sigma}{f}},$$

where the notation is as in [4]. It can be shown that the conjecture in the irreducible case is equivalent to

$$\nu_\sigma = (-1)^{f+1+\sum_{j=0}^{f-1} r_j}.$$

Let us now assume that $\rho_\ell$ is irreducible. For $v|p$ denote by $\tau_v : I_{L_v} \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$ the inertial type which is a representation of the inertia subgroup $I_{L_v}$ of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/L_v)$ with open kernel such that it can be extended to $\mathrm{Gal}(\bar{\mathbb{Q}}_p/L_v)$. denote by $\sigma(\tau_v)$ : $\mathrm{GL}_2(\mathcal{O}_{L_v}) \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$ the corresponding irreducible representation which we get by the inertial local Langlands correspondence (for more information, see appendix in [8]). With $\bar{\sigma}(\tau_v)$ we will denote the reduction modulo $p$ of some lattice in $\sigma(\tau_v)$. Note that it is not uniquely defined, but we can ignore this if we are just interested in Jordan-Holder factors of $\bar{\sigma}(\tau_v)$. Fix $\tau := \otimes_{v|p}\tau_v$ where for all $v|p$, $\tau_v$ is some inertial type such that there is at least one Diamond weight $\sigma_v \in D(\rho_v)$ appearing as a subquotient of $\bar{\sigma}(\tau_v)$. It is first conjectured in [3] that the set of weights at which $\rho$ is modular is actually $\{\otimes_{v|p}\sigma_v \mid \sigma_v \in D(\rho_v)\}$ and later proved in [18]. Using lemma 2.1.4 in [18] we get that there is a lift $\widetilde{\rho}$ of $\rho$ such that $\widetilde{\rho}\mid_{I_v}$ is isomorphic to $\tau_v$ for any $v|p$. Then we have that there is some cuspidal automorphic representation $\widetilde{\pi} \cong \otimes_v\widetilde{\pi}_v$ of $\mathrm{GL}_2(\mathbb{A}_L^f) \cong (D \otimes \mathbb{A}_L^f)^\times$ over $\bar{\mathbb{Q}}_p$ such that $\widetilde{\rho} \cong \rho_\pi$. Here $D$ denotes a definite quaternion algebra, that is $D$ is split at all finite places and compact modulo the center at all archimedean places, so that we have $(D \otimes \mathbb{A}_L^f)^\times \cong \mathrm{GL}_2(\mathbb{A}_L^f)$. Now we have that $\sigma(\tau_\ell) \subset \widetilde{\pi}_\ell \hookrightarrow \widetilde{\pi}$.

Let $\widetilde{\psi} : L^\times\backslash(\mathbb{A}_L^f)^\times \to \bar{\mathbb{Z}}_p$ such that $\widetilde{\psi} = [\omega^{-1}det(\rho)] = [\psi]$ where [ ] denotes the Teichmuller lift and $\omega$ denotes the reduction modulo $p$ of the cyclotomic character. Let $U \subset (D \otimes_L \mathbb{A}_L^f)^\times$ be an open compact subgroup such that $\psi\mid_{(U\cap\mathbb{A}_L^f)^\times} = 1$. We define $S_\psi^D(U, \bar{\mathbb{Z}}_p)$ as a $\bar{\mathbb{Z}}_p$ module consisting of functions $f : D^\times\backslash(D \otimes_L \mathbb{A}_L^f)^\times/U \to \bar{\mathbb{Z}}_p$ such that $f(ad) = \widetilde{\psi}(a)f(d)$ for any $a \in \mathbb{A}_L^f$, $d \in (D \otimes_L \mathbb{A}_L^f)^\times$. Finally we define

$S_\psi^D(\bar{\mathbb{Z}}_p) := \varinjlim_{U} S_\psi^D(U, \bar{\mathbb{Z}}_p)$. We define $S_\psi^D(\bar{\mathbb{Q}}_p)$ in the same way as $S_\psi^D(U, \bar{\mathbb{Z}}_p)$, by replacing $\bar{\mathbb{Z}}_p$ with $\bar{\mathbb{Q}}_p$. Notice that we have a modulo $p$ map $S_\psi^D(\bar{\mathbb{Z}}_p) \to S_\psi^D(\bar{\mathbb{F}}_p)$.

We have that $\sigma(\tau_\ell) \subset \widetilde{\pi}_\ell \hookrightarrow \widetilde{\pi} \subset S_\psi^D(\bar{\mathbb{Q}}_p)$ so a $\bar{\mathbb{Z}}_p$ lattice $S_\psi^D(\bar{\mathbb{Z}}_p)$ in $S_\psi^D(\bar{\mathbb{Q}}_p)$ induces a lattice $\sigma^0(\tau_\ell) \subset \sigma(\tau_\ell) \subset \widetilde{\pi}_\ell$. Breuil first conjectured in [6] that in the case when $\sigma(\tau_\ell)$ is of principal series type, the lattice $\sigma^0(\tau_\ell)$ up to homothety can be read from the Dieudonne module associated to $\sigma(\tau_\ell)$. This is proved by Gee, Emerton and Savitt in [14]. They also proved that a similar result holds in the case when $\sigma(\tau_\ell)$ is a cuspidal type (theorem 8.2.1 in [14]). More precisely, we have that $\sigma^0(\tau_\ell)$ is homothetic to $\sigma(\widetilde{\rho}_\ell) := \sum_{J \in \iota(\mathcal{P}_{\tau_\ell})} \varpi_J \sigma^0_{\iota(J)}(\tau_\ell)$. Here $\mathcal{P}_{\tau_\ell}$ denotes a set of subsets of $\{1, ..., f_\ell\}$ ($f_\ell = [L_l : \mathbb{Q}_p]$) which parametrise Jordan-Holder factors of $\bar{\sigma}(\tau_\ell)$. Also, $\sigma^0_{\iota(J)}(\tau_\ell)$ denotes a lattice in $\sigma(\tau_\ell)$ such that the cosocle of $\bar{\sigma}^0_{\iota(J)}(\tau_\ell)$ is equal to $\bar{\sigma}_{\iota(J)}(\tau_\ell)$. When $\sigma(\tau_\ell)$ is of principal series type, we have that $\sigma(\tau_\ell) = \sigma(\eta \otimes \eta') := ind_{I(\mathcal{O}_{L_\ell})}^{K(\mathcal{O}_{L_\ell})}(\eta \otimes \eta')$, where $\eta$ and $\eta'$ are some $\bar{\mathbb{Z}}_p$ characters of $k_\ell$ ($\mathcal{O}_{L_\ell}$ is the ring of integers of $L_\ell$, $k_\ell$ the residue field of $L_\ell$, $K(\mathcal{O}_{L_\ell}) := \mathrm{GL}_2(\mathcal{O}_{L_\ell})$ and $I(\mathcal{O}_{L_\ell})$ its subgroup of upper triangular matrices modulo $\pi_{L_\ell}$, where $\pi_{L_\ell}$ is the uniformizer of $\mathcal{O}_{L_\ell}$). Then the $\varpi_J\,'s$ are determind by the $\eta$ and $\eta'$ gauges of the Dieudonne module associated to $\sigma(\tau_\ell)$. Similarly, if $\sigma(\tau_\ell)$ is of cuspidal type, then we have that $\sigma(\tau_\ell) = \Theta(\eta)$, where again $\eta$ is some $\bar{\mathbb{Z}}_p$ character of $k_\ell^2$ and $\Theta(\eta)$ is as in [11] ($k_\ell^2$ is the quadratic extension of $k_\ell$). denote by $BC(\sigma(\tau_\ell))$ the base change of $\sigma(\tau_\ell)$ as in [14]. We have that $BC(\sigma(\tau_\ell)) = \sigma(\eta \otimes \eta') = ind_{I(\mathcal{O}_{L_\ell^2})}^{K(\mathcal{O}_{L_\ell^2})} \eta \otimes \eta'$ where $\eta' = \eta^{p^{f_\ell}}$ ($L_\ell^2$ the quadratic unramified extension of $L_\ell$). In this case the $\varpi_J\,'s$ can be read from the $\eta$ and $\eta'$ gauges of the Dieudonne module associated to $BC(\sigma(\tau_\ell)) = \sigma(\eta \otimes \eta')$.

In our quest of proving the conjecture in [4] by computing in $\sigma^0(\tau_\ell) \subset \sigma(\tau_\ell) \subset \widetilde{\pi}_\ell$ (where $\sigma(\tau_\ell)$ is principal series type), we were led to look into not only lattices in

$\widetilde{\pi}_\ell^{K_1(\mathcal{O}_{L_\ell})}$, but in $\widetilde{\pi}_\ell^{K_n(\mathcal{O}_{L_\ell})}$ for $n > 1$, where $K_n(\mathcal{O}_{L_\ell}) \subset K(\mathcal{O}_{L_\ell})$ is the subgroup of $K(\mathcal{O}_{L_\ell})$ of matrices congruent to identity modulo $\pi_{L_\ell}$. This together with exploring how the lattice conjecture of [6] might generalize to the setting of wildly ramified types was the motivation for the research done in this thesis. In chapter 1, we describe preliminary results and tools used in the thesis. We describe results from the representation theory of finite groups over a field of characteristic $p$ and also over the $p$-adics. We also include results from Brauer modular theory, that we make use later. In chapter 2 we give the definition of a lattice and list some of their properties in specific representations. Then we continue investigating lattices in irreducible representations over the $p$-adics and we show some of their properties. We then associate a graph to a family of lattices and we prove a few properties of the associated graphs. This includes finiteness, connectedness, a duality property and that the length of a cycle is a multiple of the number of the Jordan-Holder factors. The last section of the chapter is about computing lattices in certain representations of specific groups, looking into their properties, where some of them are derived from previous sections results. In chapter 3, we concentrate on describing lattices in representations of principal series type. Moreover, we fully describe the graphs associated to lattices in representations of tame principal series type. We continue with finding the bound for lattices in representations of principal series type. That is for a principal series type representation $\rho$ over a $p$-adic field $E$, we find a stable lattice $\Lambda \subset \rho$ and a integer $c$ such that for any stable lattice $\Lambda' \subset \rho$ up to homothety we have that $\pi_E^c \Lambda \subseteq \Lambda' \subseteq \Lambda$ (for a uniformizer $\pi_E$). We finish the chapter with giving a basis of $T$- eigenvectors of Jordan-Holder factors of $\Lambda/\pi_E\Lambda$, where $\Lambda$ is a lattice in a representation of principal series type. In the last chapter we describe the Magma code used to compute the lattices in the explicit cases. We also include the Magma code in the last section.

## 1.1 $\mathrm{GL}_2(\mathcal{O}_F)$ representations in characteristic $p$

Let $F$ be the unramified extension $\mathbb{Q}_{p^f}$ of $\mathbb{Q}_p$ of degree $f$ and let $q := p^f$. We consider continuous representations over $\bar{\mathbb{F}}_p$, which is equipped with the discrete topology. In the thesis we always consider a finite dimensional representation of a finite group $G$ unless stated otherwise.

**Theorem 1.1.1.** *Let $\pi$ be a representation of a pro-p group $G$ over a non-zero $\bar{\mathbb{F}}_p$ vector space. Then there is some non-zero element $v$ in $\pi^G$.*

*Proof.* See lemma 7 in [12]. $\square$

**Theorem 1.1.2.** *Let $\sigma$ be an irreducible representation of a group $G$ over $\bar{\mathbb{F}}_p$, where $G$ has a normal pro-p subgroup that we denote by $G_p$. Then $\sigma$ factors through $G_p$. Thus any irreducible $G$ representation over $\bar{\mathbb{F}}_p$ gives a $G/G_p$ representation over $\bar{\mathbb{F}}_p$.*

*Proof.* Since $G_p$ is normal in $G$, we have that $\sigma^{G_p}$ is stable under the $G$ action, that is $\sigma^{G_p}$ is a subrepresentation of $\sigma$. Viewing $\sigma$ as a $G_p$ representation $\sigma\ |_{G_p}$ and using theorem 1.1.1 we have that there is $0 \neq v \in \sigma^{G_p}$, so $\sigma^{G_p}$ is a non-trivial subrepresentation of $\sigma$. But since $\sigma$ is irreducible we have that $\sigma = \sigma^{G_p}$. $\square$

**Definition 1.1.3.** Let $K := \mathrm{GL}_2(\mathcal{O}_F)$. We define a weight of $K$ to be a continuous irreducible representation of $K$ over $\bar{\mathbb{F}}_p$.

**Theorem 1.1.4.** *Let $\sigma$ be a weight of $K$. Then $\sigma$ factors through $K_1$, where $K_1 \subset K$ is the subgroup of matrices which are congruent to the identity modulo $p$. So any* *continuous irreducible $K$ representation over $\bar{\mathbb{F}}_p$ gives a $GL_2(\mathbb{F}_q)$ representation over $\bar{\mathbb{F}}_p$.*

*Proof.* It follows from theorem 1.1.2. □

Let us denote by $I$ the subgroup of $K$ of upper triangular matrices mod $p$, with $\bar{T}$ the subgroup of diagonal matrices of $\mathrm{GL}_2(\mathbb{F}_q)$ and with $I_1$ the is the kernel of the projection homomorphism from $I$ to $\bar{T}$.

**Theorem 1.1.5.** *Let $\chi$ be a 1- dimensional representation of $I$ over $\bar{\mathbb{F}}_p$. Then $\chi$ factors through $I_1$. So any $I$ character over $\bar{\mathbb{F}}_p$ can be viewed as $I/I_1 \cong \bar{T}$ character over $\bar{\mathbb{F}}_p$.*

*Proof.* We have an exact sequence

$$0 \to I_1 \to I \to \bar{T} \to 0$$

where $I_1 \to I$ is the inclusion and $I \to \bar{T}$ the projection map. This gives us that $I_1$ is normal in $I$. Now using the theorem 1.1.2 we get that $\chi$ factors through $I_1$, that is $\chi^{I_1} = \chi$. □

Let us denote by $\bar{K} := \mathrm{GL}_2(\mathbb{F}_q)$ and with $\bar{I}$ the subgroup of $\bar{K}$ of upper triangular matrices.

**Corollary 1.1.6.** *Let $\chi$ be a character of $I$ over $\bar{\mathbb{F}}_p$. We have that $ind_I^K \chi \cong ind_{\bar{I}}^{\bar{K}} \chi$, where we see $ind_{\bar{I}}^{\bar{K}} \chi$ as representation of $K$ via inflation.*

*Proof.* It follows from the theorem 1.1.5. We are actually just saying that any character $\chi$ of $I$ over $\bar{\mathbb{F}}_p$ can be obtain via inflation from a character of $\bar{I}$ over $\bar{\mathbb{F}}_p$. □

In this thesis we will be often using the above theorems even without explicit reference.

**Theorem 1.1.7.** *The set of all irreducible $\bar{K}$ representations over $\bar{\mathbb{F}}_p$ is given by*

$$(Sym^{s_0}\bar{\mathbb{F}}_p^2) \otimes_{\bar{\mathbb{F}}_p} (Sym^{s_1}\bar{\mathbb{F}}_p^2)^{Frob} ... \otimes_{\bar{\mathbb{F}}_p} (Sym^{s_{f-1}}\bar{\mathbb{F}}_p^2)^{Frob^{f-1}} \otimes det^e$$

*where for all $0 \leq i \leq f-1$ we have $0 \leq s_i \leq p-1$ and $0 \leq e < p^f - 1$. Any two representations in the above set are non-equivalent.*

*Proof.* See lemma 2.16 and proposition 2.17 in [5]. $\square$

Let us denote by $N$ the normaliser of $I$ in $GL_2(F)$, which is actually the subgroup of $\mathrm{GL}_2(F)$ generated by $I$ and $\Pi := \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$.

**Theorem 1.1.8.** *Let $V$ be a $\bar{\mathbb{F}}_p$ vector space equipped with a $K$ and $N$ action which coincide on $K \cap N = I$. Then there is a unique action of $GL_2(F)$ extending the action of $K$ and $N$.*

*Proof.* See corollary 3.4 in [5].

$\square$

## 1.2 $p-$adic representations of a finite group

In this section let us denote by $E$ a finite extension of $\mathbb{Q}_p$. Let $\rho : G \to \mathrm{GL}_n(E)$ be a finite dimensional representation over $E$ of a finite group $G$ and we assume that $E$ contains all eigenvalues of action matrices of $\rho$. Let us also denote the ring of integers of $E$ with $\mathcal{O}_E$.

**Definition 1.2.1.** Let $\rho$ be as above. We define a lattice in $\rho$ to be a finitely generated $\mathcal{O}_E[G]$-submodule of $\rho$.

**Definition 1.2.2.** We define the $\chi_\rho$ character of $\rho$ as the trace of the action matrices, that is $\chi_\rho(g) := Tr(\rho(g))$.

It is a known fact that the trace of a matrix is equal to the sum of eigenvalues of the matrix, so we could define the character of a representation to be the function such that for $g \in G$, we have that $\chi_\rho(g)$ is equal to the sum of eigenvalues of $\rho(g)$. Moreover, notice that characters of $G$ are class functions. That is for any $g_1$, $g_2 \in G$ and any character $\chi$ of $G$ we have that $\chi(g_2^{-1}g_1g_2) = \chi(g_1)$.

**Theorem 1.2.3.** *We have that two finite dimensional representations of a finite group $G$ over $\mathbb{C}_p$ are isomorphic if and only if they have the same character.*

*Proof.* See theorem 14.21 in [13]. □

**Definition 1.2.4.** We define an irreducible character of a finite group to be a character corresponding to a simple representation.

**Theorem 1.2.5.** *The number of simple $E$ representations of a finite group $G$ is equal to the number of equivalence classes of $G$.*

*Proof.* See theorem 15.3 in [13]. □

**Theorem 1.2.6.** *The set of irreducible characters is a basis of the space of class functions. This means that if $\chi$ is the character of $\rho$, then there are irreducible characters $\chi_1, \dots, \chi_n$ and non-negative integers $d_1, \dots, d_n$ such that $\chi = \sum\limits_{i=1}^{n} d_i\chi_i$. Using theorem 1.2.3 and the fact that $\rho$ is semisimple (since $G$ is finite and since $E$ contains all eigenvalues of action matrices of $\rho$) we have that $\rho \cong \bigoplus\limits_{i=1}^{n} d_i\sigma_i$, where $\sigma_i$ is a simple module of $G$ corresponding to $\chi_i$.*

*Proof.* See theorem 14.17 in [13]. □

In the rest of the section we do not assume that $G$ is finite.

**Theorem 1.2.7.** *Let $H$ be a finite index subgroup of a group $G$ and let $V$ be an $H$ representation over $E$. We consider the induced representation $ind_H^G V := E[G] \otimes_{E[H]} V$ and the coinduced representation $coind_H^G V := Hom_{E[H]}(E[G], V)$. We have that $ind_H^G V \cong coind_H^G V$.*

*Proof.* denote by $f_{Hg,v} \in Hom_{E[H]}(E[G], V)$ the unique $H$-linear map supported on $Hg$ such that $f_{Hg,v}(g) = v$. Let us define a map

$$\phi : E[G] \otimes_{E[H]} V \to Hom_{E[H]}(E[G], V)$$

such that $\phi(g \otimes v) = f_{Hg^{-1},v}$. $\phi$ is $G$ invariant since for any $g$, $g_1$, $g_2 \in G$ and $v \in V$ we have

$$\begin{aligned}
(g_1(\phi(g \otimes v)))(g_2) &= g_1(f_{Hg^{-1},v})(g_2) \\
&= (f_{Hg^{-1},v})(g_2 g_1) \\
&= f_{Hg^{-1}g_1^{-1},v}(g_2) \\
&= \phi(g_1 g \otimes v)(g_2) \\
&= \phi(g_1(g \otimes v))(g_2).
\end{aligned}$$

Moreover it is not hard to see that $\phi$ is injective and surjective, which finishes the proof. □

**Theorem 1.2.8.** *Let $H$ be a finite index subgroup of a group $G$ and let $V$ be an $H$ representation over $E$. We have that $ind_H^G(V^\vee) \cong (ind_H^G V)^\vee$.*

*Proof.* We know that $ind_H^G V = E[G] \otimes_{E[H]} V$, so we have that

$$(ind_H^G V)^\vee = \mathrm{Hom}_E(E[G] \otimes_{E[H]} V, E),$$

where for any $f \in \mathrm{Hom}_E(E[G] \otimes_{E[H]} V, E)$ and $g \in G$ we have $(gf)(x) = f(g^{-1}x)$. Let us denote by $\phi$ the map

$$\phi : \mathrm{Hom}_{E[H]}(E[G], \mathrm{Hom}_E(V, E)) \xrightarrow{\sim} \mathrm{Hom}_E(E[G] \otimes_{E[H]} V, E).$$

obtained by sending $f \in \mathrm{Hom}_{E[H]}(E[G], \mathrm{Hom}_E(V, E))$ to $f' = \phi(f) \in \mathrm{Hom}_E(E[G] \otimes_{E[H]} V, E)$, such that $f'(g \otimes v) = (f(g^{-1}))(v)$. Notice that $\mathrm{Hom}_E(V, E)$ is isomorphic to $V^\vee$ and for any $f \in \mathrm{Hom}_E(V, E)$ and $h \in H$ we have that $(hf)(x) = f(h^{-1}x)$. We have that $\phi$ is $G$-invariant since for any $g_1 \in G$ and any $g \otimes v \in E[G] \otimes_{E[H]} V$ we have

$$\begin{aligned}
(g_1(\phi(f)))(g \otimes v) &= (\phi(f))(g_1^{-1} g \otimes v) \\
&= (f(g^{-1} g_1))(v) \\
&= (g_1 f(g^{-1}))(v) \\
&= \phi(g_1 f)(g \otimes v).
\end{aligned}$$

It is not hard to see that $\phi$ is injective and surjective, which gives that $\phi$ is a $G$-equivariant isomorphism. Since $\mathrm{Hom}_E(V, E)$ is the dual of $V$ we have that

$$\mathrm{Hom}_{E[H]}(E[G], \mathrm{Hom}_E(V, E)) = coind_H^G(V^\vee).$$

Using 1.2.7 we have that

$$\mathrm{Hom}_{E[H]}(E[G], \mathrm{Hom}_E(V, E)) \cong ind_H^G(V^\vee).$$

This finishes the proof.

**Definition 1.2.9.** Let $F$ be a finite extension of $\mathbb{Q}_p$, $\mathcal{O}_F$ its ring of integers, $\pi_F$ its uniformizer. Let $G := \mathrm{GL}_2(\mathcal{O}_F)$ and let $I_n \subset G$ be the subgroup of $G$ of upper triangular matrices modulo $\pi_F^n$. Let $\chi : I_n \to E$ be a character of $I_n$. We define a $p$-adic representation of principal series type to be $ind_{I_n}^G \chi$. In the case when $n = 1$ we call it of tame type, otherwise it is of non-tame type.

$\square$

## 1.3 Brauer modular theory

In this section let $E$ be some finite extension of $\mathbb{Q}_p$, $\mathcal{O}_E$ its ring of integers, $\pi_E$ its uniformiser and $k_E$ its residue field. Let $\rho_p : G \to \mathrm{GL}_n(k_E)$ be a finite dimensional continuous representation of some finite group $G$ and let us denote by $\mu_E \subset E$ the union of all $m$-th roots of unity contained in $E$, where $m$ runs through all integers coprime to $p$. So we can fix an isomorphism $k_E^\times \to \mu_E$. We define $G_{reg}$ to be the subset of $G$, consisting of elements of order coprime to $p$. We have that for any $g \in G_{reg}$, the order of $\rho(g)$ is coprime to $p$. Hence and because $\mu_E \subset E$ the matrix $\rho(g)$ is diagonalisable and the eigenvalues of $\rho(g)$ are $n$-th roots of unity, where $n$ is the order of $g$.

**Definition 1.3.1.** We define the modular chararter $\chi_{\rho_p} : G_{reg} \to E$ of $\rho_p$ to be the function defined by $\chi_{\rho_p}(g)$ is the sum of eigenvalues of $\rho_p(g)$, where $g \in G_{reg}$.

Let us denote by $\rho : G \to \mathrm{GL}_n(E)$ a finite dimensional representation of a finite group $G$. We assume that $E$ is such that all absolutely irreducible $p$-adic representations (that is representations that stay irreducible after any extension of $E$) of $G$ are realised over $E$ and all eigenvalues of their action matrices are contained in $E$. Let $\Lambda$

be a finitely generated $\mathcal{O}_E[G]$ submodule of $\rho$ such that $\rho = \Lambda \otimes_{\mathcal{O}_E} E$, where we see $\mathcal{O}_E[G]$ module $\Lambda$ as a representation of $G$ over $\mathcal{O}_E$. We know that such $\Lambda$ exists since if $E_1, ..., E_n$ is a basis of $\rho$ over $E$, we could take for $\Lambda$ to be the $\mathcal{O}_E[G]$ submodule of $\rho$ generated by $E_1, ..., E_n$. Define $\bar{\rho} := \Lambda/\pi_E\Lambda$. Since $k_E \cong \mathcal{O}_E/\pi_E\mathcal{O}_E$ we can view $\bar{\rho}$ as a representation over $k_E$. Let us fix an $\mathcal{O}_E$ basis $\beta$ of $\Lambda$ and denote it with $E_1$, ..., $E_n$. For $g \in G_{reg}$ we denote the action matrix of $g$ in $\rho$ with respect to the basis $\beta$ with $M_g \subset \mathrm{GL}_n(E)$.

**Theorem 1.3.2.** *Let $\rho$, $\Lambda$ and $\bar{\rho}$ be as above. We have that $\chi_\rho \mid_{G_{reg}} = \chi_{\bar{\rho}}$.*

*Proof.* We have that the action matrix of $g$ in $\bar{\rho}$ is $\bar{M}_g \subset \mathrm{GL}_n(k_E)$, where the corresponding entries of the matrix $\bar{M}_g$ are the corresponding entries of the matrix $M_g$ modulo $\pi_E\mathcal{O}_E$. This means that the eigenvalues of $\bar{M}_g$ are the eigenvalues of $M_g$ modulo $\pi_E\mathcal{O}_E$. But since the order of $g$ is coprime to $p$, we have that the order of $M_g$ is coprime to $p$ as well. This gives us that the eigenvalues of $M_g$ are $n$-th roots of unity, where $n$ is coprime to $p$. That is its eigenvalues are elements of $\mu_E$, which gives us that the set of eigenvalues of $M_g$ is equal to the set of eigenvalues of $\bar{M}_g$, via the fixed isomorphim $k_E^\times \to \mu_E$. Finally, this implies that $\chi_{\bar{\rho}}(g) = \chi_\rho(g)$. □

**Definition 1.3.3.** We define an irreducible modular character of a group $G$ to be the modular character of a simple $k_E[G]$-module.

**Theorem 1.3.4.** *Let $k_E$ be as in the beginning of the section. We have that the number of isomorphism classes of irreducible $k_E[G]$-modules is equal to the number of conjugacy classes in $G_{reg}$. Similarly as in the characteristic 0 case, irreducible modular characters make a basis of the space of class functions on $G_{reg}$.*

*Proof.* See corollary 3 in [17]. □

**Definition 1.3.5.** Let $G$ be a group, $\rho$ a $G$-module. By a Jordan-Holder factor of $\rho$ we mean an irreducible $G$-module $J$ such that $J \cong M/M'$ where $M' \subset M$ are $G$-stable submodules of $\rho$.

**Theorem 1.3.6.** *Two representations over $k_E$ have the same Jordan-Holder factors (counted with multiplicities) if and only if they have the same modular character.*

*Proof.* See corollary 1 in [17]. $\square$

**Theorem 1.3.7.** *Let $\chi$ be the modular character of $\rho_p$. We have that $\chi = \sum\limits_{i=1}^{n} d_i \chi_i$, where $\chi_i$'s are irreducible modular characters and $d_i$ are non-negative integers. The Jordan-Holder factors of $\rho$ are $(\sigma_i)_{1 \le i \le n}$, where the multiplicity of $\sigma_i$ is given by $d_i$.*

*Proof.* It follows from 1.3.4 and 1.3.6. $\square$

**Theorem 1.3.8.** *Let $\chi$ be the character of $\rho$ and $\bar{\chi}$ the character of $\bar{\rho}$. From 1.3.2, we have that $\chi \mid_{G_{reg}} = \bar{\chi}$. Let $\chi_i$'s be irreducible modular characters and $d_i$ non-negative integers such that $\chi \mid_{G_{reg}} = \bar{\chi} = \sum\limits_{i=1}^{n} d_i \chi_i$. We have that the Jordan-Holder factors of any lattice in $\rho$ are given by $\{\sigma_i\}_{1 \le i \le n}$, where the multiplicity of $\sigma_i$ is given by $d_i$ (here we see a lattice in $\rho$ as a $\mathcal{O}_E[G]$ module).*

*Proof.* See 1.3.7 and 1.3.8. $\square$

**Definition 1.3.9.** Let $\rho_1, ..., \rho_m$ be the absolutely irreducible $p$-adic representations of $G$, $\chi_1, ..., \chi_n$ their corresponding characters (by a $p$-adic representation we mean a representation over an extension of $\mathbb{Q}_p$). Let $\zeta_j$ be the irreducible modular characters of $G$ and $d_{j,i}$ non-negative integers such that $\chi_i \mid_{G_{reg}} = \sum\limits_{j=1}^{m} d_{j,i} \zeta_j$, where $1 \le i \le n$, $1 \le j \le m$. We define the decomposition matrix of $G$ to be the matrix $D := (d_{j,i})_{\substack{1 \le j \le m \\ 1 \le i \le n}}$.

# Chapter 2

# Lattices in irreducible $p$ - adic representations

Let $E$ be a $p$-adic field, $V$ a $n$-dimensional vector space over $E$ and $\rho : G \to Aut_E(V)$ an absolutely irreducible representation of a finite group $G$, unless stated otherwise. Let us denote by $\mathcal{O}_E$ the ring of integers of $E$ and with $\pi_E$ a uniformiser. Having in mind that $Aut_E(V) \overset{\sim}{=} \mathrm{GL}_n(E) \subset \mathrm{M}_n(E)$ and that $\rho$ is absolutely irreducible, we have that the $E$ subalgebra of $\mathrm{M}_n(E)$ generated by $im(\rho)$ is actually the whole $\mathrm{M}_n(E)$ (see the theorem 2.5 in [10]). denote by $\Lambda_\rho$ the $\mathcal{O}_E$ order in $\mathrm{M}_n(E)$ generated by $im(\rho)$, that is the $\mathcal{O}_E$ module in $\mathrm{M}_n(E)$ generated by $im(\rho)$.

## 2.1 Properties of lattices in specific cases

In this section we mostly rely on Plesken's book [19].

**Definition 2.1.1.** $\Lambda_\rho$ is called a graduated order if there exist orthogonal primitive idempotents $\epsilon_1, .., \epsilon_t$ in $\Lambda_\rho$ such that $1 = \epsilon_1 + ... + \epsilon_t$ and $\epsilon_i \Lambda_\rho \epsilon_i$ is a maximal order in $\epsilon_i \mathrm{M}_n(E) \epsilon_i$, for $i = 1, ..., t$.

**Definition 2.1.2.** A set $\alpha$ of $\mathcal{O}_E$ - lattices is an admissible system if:

- for $L_1, L_2 \in \alpha$, we have $L_1 \cap L_2 \in \alpha$ and $L_1 + L_2 \in \alpha$;
- for $L \in \alpha$ and $a \in \mathbb{Z}$ we have $\pi_E^a L \in \alpha$;
- there is a constant $\delta \in \mathbb{N}$ such that for any two $L_1, L_2 \in \alpha$, there exists $a \in \mathbb{Z}$ with $L_1 \pi_E^\delta \subset L_2 \pi_E^a \subset L_1$.

Using the theorem 2.2.4 we have that the set of non-zero lattices $\Lambda_\rho$ in $V$ forms an admissible system.

**Definition 2.1.3.** An admissible system of $\mathcal{O}_E$ lattices in $V$ is said to have a system of compatible bases, if there is an $\mathcal{O}_E$ basis $B$ of $V$ such that one obtains an $\mathcal{O}_E$ basis for each $L' \in \alpha$ by multiplying the basis vectors in $B$ with elements of $E$. That is the transformation matrix can be chosen to be diagonal.

**Definition 2.1.4.** For a set of lattices $\mathcal{F}$ we say that it is distributive if for any $\Lambda_1$, $\Lambda_2$, $\Lambda_3 \in \mathcal{F}$ we have that $(\Lambda_1 + \Lambda_2) \cap \Lambda_3 = (\Lambda_1 \cap \Lambda_3) + (\Lambda_2 \cap \Lambda_3)$.

**Theorem 2.1.5.** *Let $\alpha$ be an admissible system of $\mathcal{O}_E$ lattices in the $E$ vector space $V$. Then $\alpha$ has a system of compatible bases if and only if $\alpha$ is distributive.*

*Proof.* See proposition $II.7$ in [19]. $\square$

**Theorem 2.1.6.** *The following are equivalent:*

- *$\Lambda_\rho$ is a graduated order;*
- *$\Lambda_\rho$ is the intersection of the maximal orders containing it in $M_n(E)$ and the non zero $\Lambda_\rho$ lattices in $V$ form a distributive family (with respect to the operations $+$ and $\cap$);*

- *for each lattice $L$ in $V$, the $\Lambda_\rho$ decomposition factors $S_1, ..., S_t$ of the $\Lambda_\rho$ (torsion) module $L/\pi_E L$, all have multiplicity 1 in a $\Lambda_\rho$ - composition series of $L/\pi_E L$ and $\Lambda_\rho$ is the intersection of the maximal orders containing it in $M_n(E)$.*

*Proof.* See proposition $II.8$ in [19]. □

## 2.2 Associating graphs to irreducible representations and their properties

Let $G$ be a finite group, $E$ a finite extension of $\mathbb{Q}_p$, $V$ a finite dimensional vector space over $E$ and $\rho : G \to Aut_E V$ an absolutely irreducible representation. denote by $\mathcal{O}_E$ the ring of integers of $E$, $\pi_E$ a uniformiser and $k_E$ its residue field. As before, a lattice is a finitely generated $\mathcal{O}_E[G]$-submodule of $V$.

**Definition 2.2.1.** We define a homothety class of a lattice $\Lambda$ to be the set $[\Lambda]$ of lattices such that for any $\Lambda' \in [\Lambda]$ we have $\Lambda' = c\Lambda$ for some $c \in E^\times$.

Having in mind that $\rho$ is irreducible and using Schur's lemma (lemma 2.1 in [16]) we have that two lattices are homothetic if and only if they are irreducible.

**Definition 2.2.2.** We associate to $\rho$ a directed graph constructed as follows:

- a vertex is a homothety class of lattices in $V$;
- a directed edge from $\Lambda$ to $\Lambda'$ exists if there exists $a \in \mathbb{Z}$ such that $\pi_E \Lambda \subseteq \pi_E^a \Lambda' \subseteq \Lambda$ and the quotient $\Lambda/\pi_E^a \Lambda'$ is irreducible.

Moreover we have a labeled version of this graph, where an edge is labeled according to the isomorphism class of $\Lambda/\Lambda'$. We denote the labelled graph with $\mathcal{G}$.

In this section we will show some properties of these graphs.

**Lemma 2.2.3.** *Let $\Omega' \subset \Omega$ be $\mathcal{O}_E$ lattices in $V$. Then there are only finitely many $\mathcal{O}_E$-lattices $\Lambda$ such that $\Omega' \subset \Lambda \subset \Omega$.*

*Proof.* Notice that there is a bijection between the lattices $\Lambda$ such that $\Omega' \subset \Lambda \subset \Omega$ and the $\mathcal{O}_E$ -submodules of the quotient $\Omega/\Omega'$ , given by $\Lambda \mapsto \Lambda/\Omega'$. But $\Omega/\Omega'$ is finite so has only finitely many $\mathcal{O}_E$-submodules. $\square$

**Theorem 2.2.4.** *Let $\rho : G \to Aut_E(V)$ be an irreducible finite dimensional representation of a finite group $G$, not necessarily absolutely irreducible. Then the number of lattices in $V$ is finite up to homothety.*

*Proof.* Let us assume the contrary, that is assume that there is a family of non-homothetic lattices $\{V_i\}_{i\in\mathbb{N}}$ where $\mathbb{N}$ is the ring of positive integers. Without loss of generality, we can assume that for any $i \in \mathbb{N}$ and $i > 1$ we have $V_i \subset V_1$ and $V_i \not\subset \pi_E V_1$. For all $i > 1$ let us fix $E_i$ such that $E_i \in V_i$, $E_i \notin \pi_E V_i$, $E_i \in V_1$ and $E_i \notin \pi_E V_1$, which is possible because of the way how we scaled the lattice $V_i$. Let us show by induction that we can construct the family of lattices $\{V_i\}_{i\in\mathbb{N}}$ and sequence $(E_i)_{i\in\mathbb{N}}$ such that $E_i - E_j \in \pi_E^i V_1$ for all $i, j \in I$, $i < j$ (by taking an infinite sub-family of the starting one). Since $V_1/\pi_E^2 V_1$ is finite, we have that there is an infinite $I_2 \subset \mathbb{N} \setminus \{1\}$ such that $E_i - E_j \in \pi_E^2 V_1$ for all $i, j \in I$. So after renaming lattices, we obtain that $E_2 - E_j \in \pi_E^2 V_1$ for all $j > 2$. Define $E_1 := E_2$. Let us now assume that for any $i < k$ and any $j > i$ we have that $E_i - E_j \in \pi_E^i V_1$. Now we need to show that this implies that we can reconstruct the family by potentially changing only $V_j$ for $j \geq k$ such that for any $j > k$ we have that $E_k - E_j \in \pi_E^k V_1$. But again, since $V_1/\pi_E^k V_1$ is finite, we have that there is an infinite $I_k \subset \mathbb{N} \setminus \{1, 2, ..., k-1\}$ such that

$E_i - E_j \in \pi_E^k V_1$ for all $i, j \in I$. So this implies that we can rename lattices $V_i$ for $i \geq k$ such that for any $j > k$ we have that $E_k - E_j \in \pi_E^k V_1$. This completes the induction.

Let us now construct a new familly of non-homothetic lattices $\{\Lambda_i\}_{i\in\mathbb{N}}$. Let us define $\Lambda_1 := V_1$ and $\Lambda_i = (\Lambda_{i-1} \cap V_{i-1}) + \pi_E^{i-1}\Lambda_1$. Notice that since $(\Lambda_{i-1} \cap V_{i-1}) \subset \Lambda_{i-1}$ and $\pi_E^{i-1}\Lambda_1 \subset \pi_E^{i-2}\Lambda_1 \subset \Lambda_{i-1}$ we have that $\Lambda_i \subset \Lambda_{i-1}$. Also, by induction we will show that $E_i \in \Lambda_i$. We have $E_1 \in V_1 = \Lambda_1$. Let us assume that $E_{i-1} \in \Lambda_{i-1}$. We have that $E_i - E_{i-1} \in \pi_E^{i-1} V_1 = \pi_E^{i-1}\Lambda_1$ and $E_{i-1} \in \Lambda_{i-1} \cap V_{i-1}$ which implies that $E_i \in \Lambda_i$. This completes the induction.

Notice that the sequence $(E_n)_{n\in\mathbb{N}}$ is Cauchy sequence so converges, that is we have that there is some $E = \lim_{\vec{n}} E_n \in \bigcap_i (\Lambda_i)$ and $E \neq 0$ (since $E_i \notin \pi_E V_1$) which gives us that the sub lattice $\bigcap_i (\Lambda_i)$ of $\Lambda_1$ is non-zero. But applying the lemma 2.2.3 again, we arrive in contradiction with the fact that $\Lambda_i \neq \Lambda_j$ for all $i \neq j$. This finishes our proof. $\square$

**Theorem 2.2.5.** *$\mathcal{G}$ is finite.*

*Proof.* It follows from 2.2.4. $\square$

Notice that the theorem above follows as well from the local version of the Jordan-Zassenhaus theorem (theorem 24.7 in [9]).

**Theorem 2.2.6.** *$\mathcal{G}$ is strongly connected.*

*Proof.* Let $\Lambda$ and $\Lambda'$ be representatives of two nodes in $\mathcal{G}$. After rescaling we can assume that $\Lambda' \subset \Lambda$. Let us denote by $\Lambda_1, ..., \Lambda_n$ a Jordan-Holder filtration of $\mathcal{O}_E$ module $\Lambda/\Lambda'$, such that $\Lambda_1 \subset ... \subset \Lambda_n$. Let $\widetilde{\Lambda}_i$ be the lift of $\Lambda_i$ in $\Lambda$. We have that

$\Lambda' \subset \widetilde{\Lambda}_1 \subset .. \subset \Lambda$ such that the quotient of each subsequent ones is irreducible which gives us that there is a directed path from $\Lambda$ to $\Lambda'$. □

**Theorem 2.2.7.** *Any cycle in $\mathcal{G}$ is made of a multiple of the set of edges labeled with Jordan-Holder factors of $\Lambda/\pi_E\Lambda$ where $\Lambda$ is a lattice in $\rho$.*

*Proof.* Let $\omega$ be a cycle containing the node $[\Lambda]$, where with $[\Lambda]$ we represent the class of lattices. This means that there are nodes $[\Lambda_1], ..., [\Lambda_n]$ such that $[\Lambda] \to [\Lambda_1] \to ... \to [\Lambda_n] \to [\Lambda]$, where arrows are representing the corresponding edges. This means that there are class representatives $\Lambda_i$ and some integer $c_\Lambda$ such that $\pi_E^{c_\Lambda}\Lambda \subset \Lambda_1 \subset ... \subset \Lambda_n \subset \Lambda$, $\Lambda_{i+1}/\Lambda_i$, $\Lambda_1/\pi_E^{c_\Lambda}\Lambda$ and $\Lambda/\Lambda_n$ are irreducible. But then those quotients are representing Jordan-Holder's factors of $\Lambda/\pi_E^{c_\Lambda}\Lambda$, which is just a $c_\Lambda$ multiple of Jordan-Holder's factors of $\Lambda/\pi_E\Lambda$. □

**Definition 2.2.8.** Given a graph $\mathcal{G}$ we define its dual $\mathcal{G}^\vee$ by reversing the arrows and with replacing the label of the arrow labeled by a Jordan-Holder factor $\sigma$ by reversed arrow labeled with $Hom_{k_E}(\sigma, k_E)$.

**Theorem 2.2.9.** *$\mathcal{G}^\vee$ corresponds to the graph associated to $\rho^\vee$.*

*Proof.* denote by $\mathcal{G}_{\rho^\vee}$ the graph associated to $\rho^\vee$. For any $\mathcal{O}_E[G]$ stable lattice $\Lambda$ in $\rho$, we have that $\Lambda^\vee := Hom_{\mathcal{O}_E}(\Lambda, \mathcal{O}_E)$ is a $\mathcal{O}_E[G]$ stable lattice in $\rho^\vee$. Having in mind the finitness result of 2.2.5 and that $\Lambda_1 \cong \Lambda_2$ if and only if $\Lambda_1^\vee \cong \Lambda_2^\vee$, we have that the map $\Lambda \mapsto \Lambda^\vee$ is a bijection between classes of $\mathcal{O}_E[G]-$ stable lattices in $\rho$ and in $\rho^\vee$. Let $[\Lambda_1], [\Lambda_2]$ be two classes of $\mathcal{O}_E[G]-$ stable lattices in $\rho$, such that there is an arrow from $[\Lambda_1]$ to $[\Lambda_2]$. This means that there are representatives of classes $\Lambda_1, \Lambda_2$ such that $\pi_E\Lambda_1 \subset \Lambda_2 \subset \Lambda_1$ and $\sigma := \Lambda_1/\Lambda_2$ is irreducible. This is equivalent to saying that $0 \to \Lambda_2 \to \Lambda_1 \to \Lambda_1/\Lambda_2 \to 0$ is exact. But then $0 \to \Lambda_1^\vee \to \Lambda_2^\vee \to Hom_{\mathcal{O}_E}(\Lambda_1/\Lambda_2, \mathcal{O}_E) \to 0$ is exact too and $Hom_{\mathcal{O}_E}(\Lambda_1/\Lambda_2, \mathcal{O}_E) \cong \sigma^\vee$. □

**Theorem 2.2.10.** *Let $\mathcal{G}^\vee$ be as defined in 2.2.8. Let us also assume that $\rho$ is self dual up to a character, that is there is a character $\chi$ such that $\rho \cong \rho^\vee \otimes \chi$. We have that $\mathcal{G}^\vee \cong \mathcal{G}$.*

*Proof.* Let us first notice that $\Lambda$ is a stable lattice in $\rho$ if and only if $\Lambda_{\chi^{-1}}$ is a stable lattice in $\rho^\vee$, where $G$ acts on $\Lambda_{\chi^{-1}}$ via twist by $\chi^{-1}$. This gives us that $\mathcal{G}^\vee \cong \mathcal{G}$. □

## 2.3 Explicit cases

In this section we compute lattices in certain irreducible $p-$adic representations and draw their graphs. For two lattices which have an edge between, we will distinguish their edge labels using colours, that is two quotients are isomorphic if an only if the corresponding edges are of the same colour.

### 2.3.1 The 2-dimensional irreducible representation of the dihedral group $D_8$

Let $D_8$ be the dihedral group of order 8. It is a non-abelian group and since it is a $p$-group its 2-Sylow subgroup is the whole group. Thus it has a non-abelian normal 2-Sylow subgroup. $D_8$ has a presentation $< \tau, \sigma | \tau^{-1}\sigma\tau = \sigma^{-1}, \tau^2 = \sigma^4 = e >$. Let $\rho : D_8 \to \mathrm{GL}_2(E)$ be its unique irreducible 2-dimensional representation given by

$$\sigma \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tau \mapsto \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

where we take $E$ to be some finite extension of $\mathbb{Z}_2$. Let us now calculate all the $\mathcal{O}_E$ lattices in $\rho$.

Using the theorem 1.1.2 we have that all the lattices in $\rho$ have repeated Jordan Holder factors of dimension 1 (they are actually all the trivial representation). Let $e_1 = [1, 0]$ and $e_2 = [0, 1]$ be the standard basis and let $\Lambda = \mathcal{O}_E e_1 \oplus \mathcal{O}_E e_2$. We have that $\mathit{Stab}\ \Lambda = \mathrm{GL}_2(\mathcal{O}_E)$ (where by $\mathit{Stab}\ \Lambda$ we denote the set of all matrices $g \in \mathrm{GL}_2(E)$ such that $g\Lambda = \Lambda$) and all other lattices are described by $g\Lambda$ where $g \in \mathrm{GL}_2(E)$. So we now look for $g \in \mathrm{GL}_2(E)$ such that

$$\sigma, \tau \in \mathit{Stab}\ g\Lambda = g\mathit{Stab}\ \Lambda g^{-1} = g\mathrm{GL}_2(\mathcal{O}_E)g^{-1}. \tag{2.3.1}$$

Let us denote by $\mathrm{B} \subset \mathrm{GL}_2(E)$ the subgroup of $\mathrm{GL}_2(E)$ of upper triangular matrices. Since $\mathrm{GL}_2(E) = \mathrm{B}\ \mathrm{GL}_2(\mathcal{O}_E)$ (see Iwasawa decomposition in [16]) and our lattice $\Lambda$ is $\mathrm{GL}_2(\mathcal{O}_E)$ stable, it suffices to find all $g \in \mathrm{B}$ which satisfy the condition (2.3.1). Let $g = \begin{pmatrix} a & -b \\ 0 & d \end{pmatrix}$. Then our condition (2.3.1) is equivalent to

$$\frac{b}{d},\ \frac{a}{d},\ \frac{b^2}{ad} + \frac{d}{a},\ \frac{b^2}{ad} - \frac{d}{a}\ \in \mathcal{O}_E. \tag{2.3.2}$$

Let us denote by $\pi_E$ the uniformiser of $\mathcal{O}_E$, $e_E$ the ramification index so that $2u = \pi_E^{e_E}$, for some unit $u \in \mathcal{O}_E^\times$. Then the condition (2.3.2) implies that $a = Ad$, $b = Bd$ where $A,\ B \in \mathcal{O}_E$. Then we have that $\frac{B^2}{A} + \frac{1}{A} \in \mathcal{O}_E$ and $\frac{B^2}{A} - \frac{1}{A} \in \mathcal{O}_E$. So all the $\mathcal{O}_E$ lattices up to homothety inside $\rho$ stable under the $\rho$ action are given by

$$\mathcal{O}_E(Ae_1) \oplus \mathcal{O}_E(Be_1 + e_2) \tag{2.3.3}$$

such that $A$, $B$, $\frac{B^2}{A} + \frac{1}{A}$ and $\frac{B^2}{A} - \frac{1}{A} \in \mathcal{O}_E$. If $B = 0$ we get that $A \in \mathcal{O}_E^\times$, so the stable lattice in this case looks like

$$\mathcal{O}_E e_1 \oplus \mathcal{O}_E e_2. \tag{2.3.4}$$

Otherwise, we can assume that $val_E(A) > val_E(B)$. The conditions on $A$ and $B$ gives us that $val_E(B^2-1) \geq val_E(A)$ and that $val_E(B^2+1) \geq val_E(A)$. So having in mind our assumption we get that $val_E(B) = 0$. Also, we get that $val_E(2) = val_E((B^2+1)-(B^2-1)) \geq min\{val_E(B^2+1), val_E(B^2-1)\} \geq val_E(A)$. So we can assume that $A = \pi_E^i$, where $i$ ranges $0 \leq i \leq e_E$. Let us now fix $A = \pi_E^i$ for some $0 \leq i \leq e_E$ and see what the possibilities for $B$ are. Notice that the lattices $\Lambda_1 = \mathcal{O}_E(Ae_1) \oplus \mathcal{O}_E(Be_1+e_2)$ and $\Lambda_2 = \mathcal{O}_E(Ae_1) \oplus \mathcal{O}_E(B'e_1+e_2)$ are the same if and only if $\frac{B-B'}{A} \in \mathcal{O}_E$. Given that $\frac{B^2}{A}+\frac{1}{A}$ and $\frac{B^2}{A}-\frac{1}{A} \in \mathcal{O}_E$ (which is equivalent to $\frac{B^2}{A}-\frac{1}{A} \in \mathcal{O}_E$, since $\frac{2}{A} \in \mathcal{O}_E$) we get that the only possible lattices are

$$\mathcal{O}_E(\pi_E^i e_1) \oplus \mathcal{O}_E(e_1+e_2) \tag{2.3.5}$$

where $0 \leq i \leq e_E$ and

$$\mathcal{O}_E(\pi_E^i e_1) \oplus \mathcal{O}_E((1+c)e_1+e_2) \tag{2.3.6}$$

where $0 < i \leq e_E$ and $c$ runs over representatives of $\pi_E^{\lceil \frac{i}{2} \rceil}\mathcal{O}_E/\pi_E^i\mathcal{O}_E$.

Let us draw the graph in the case when $e_E = 2$. denote by $\Lambda_1 := \mathcal{O}_E e_1 \oplus \mathcal{O}_E(e_1+e_2)$, $\Lambda_2 := \mathcal{O}_E(\pi_E e_1) \oplus \mathcal{O}_E(e_1+e_2)$, $\Lambda_3 := \mathcal{O}_E(\pi_E^2 e_1) \oplus \mathcal{O}_E((1+\pi_E)e_1+e_2)$ and $\Lambda_4 := \mathcal{O}_E(\pi_E^2 e_1) \oplus \mathcal{O}_E(e_1+e_2)$. The graph is shown in figure 2.1, where the nodes are denoted with the indices of the lattices.

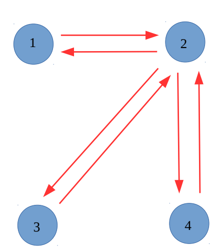

**Figure 2.1:** Dihedral group

In the thesis we are interested in describing all stable lattice in a $p$-adic representation (representation over some extension over $\mathbb{Q}_p$). We are looking into if there is some nice description of stable lattices in examples that we are studying, so we are attempting to investigate if there is a system of compatible basis for families of stable lattices in the examples. Using theorem 2.1.5 we have that a family of admissible lattices has a system of compatible basis if and only if it is distributive, so we mainly end up investigating the distributivity property. Using the theorem 2.1.6 we know that in any $p$-group any non-trivial irreducible representation will fail either the distributive property or the property of being intersection of all maximal ideals containing it. This is why in our examples we attempted to invesigate the property of being intersection of all maximal ideals containing it, but we did not manage to find out any effective method for doing this, so we are investigating the distributivity directly. Notice that for $E = \mathbb{Q}_2$ the lattices have the distributive property since they're well-ordered (and the compatible basis is $e_1$, $e_1 + e_2$). For families of bigger ramification we will show that when $E$ has higher ramification degree that is not the case. Let us look into the cases when ramification $e_E \geq 9$. Recall that the lattices described above have repeated Jordan-Holder factors. We have the following result:

**Lemma 2.3.1.** *There is no system of compatible basis for the family of lattices described above 2.3.5, 2.3.6 in case when $e_E > 1$.*

*Proof.* Using theorem 2.1.5 we have that a family of admissible lattices has a system of compatible basis if and only if it is distributive. For $1 < e_E \leq 9$ we check that distributivity property is not satisfied by running the code 4.3.3. Let us now assume that $e_E > 9$. We will show that there is no system of compatible basis for the family of lattices described above, by giving the explicit stable lattices $\Lambda_1$, $\Lambda_2$ and $\Lambda_3$ for which the distributive property fails. Using the transformation $E_1 := e_1 + e_2$ and $E_2 := e_1$ on the basis of stable lattices, we consider lattices $\Lambda_1 := \mathcal{O}_E(\pi_E^9 e_1) \oplus \mathcal{O}_E((1+\pi_E^5)e_1 + e_2) = \mathcal{O}_E(\pi_E^9 e_1 - \pi_F^4((1+\pi_E^5)e_1 + e_2)) \oplus \mathcal{O}_E(\pi_E^5 e_1 + (e_1 + e_2)) = \mathcal{O}_E(\pi_E^4 E_1) \oplus \mathcal{O}_E(\pi_E^5 E_2 + E_1)$, $\Lambda_2 := \mathcal{O}_E(\pi_E^9 e_1) \oplus \mathcal{O}_E((1+\pi_E^7)e_1 + e_2) = \mathcal{O}_E(\pi_E^9 e_1 - \pi_F^2((1+\pi_E^7)e_1 + e_2)) \oplus \mathcal{O}_E(\pi_E^7 e_1 + (e_1 + e_2)) = \mathcal{O}_E(\pi_E^2 E_1) \oplus \mathcal{O}_E(\pi_E^7 E_2 + E_1)$ and $\Lambda_3 := \mathcal{O}_E(\pi_E^9 e_1) \oplus \mathcal{O}_E(e_1 + e_2) = \mathcal{O}_E(E_1) \oplus \mathcal{O}_E(\pi_E^9 E_2)$. We show that $\pi_E^7 E_2 + E_1 \in \Lambda_2 \cap (\Lambda_1 + \Lambda_3)$ and $\pi_E^7 E_2 + E_1 \notin (\Lambda_2 \cap \Lambda_1) + (\Lambda_2 \cap \Lambda_3)$. Let us first show that $\pi_E^7 E_2 + E_1 \in \Lambda_2 \cap (\Lambda_1 + \Lambda_3)$. We have that $\pi_E^7 E_2 + E_1 \in \Lambda_2$, so it is enough to show that $\pi_E^7 E_2 + E_1 \in (\Lambda_1 + \Lambda_3)$. We have that $\pi_E^2(E_1 + \pi_E^5 E_2) \in \Lambda_1$ and $(1 - \pi_E^2)E_1 \in \Lambda_3$, so $\pi_E^7 E_2 + E_1 \in \Lambda_1 + \Lambda_3$.

Let us finally show that $\pi_E^7 E_2 + E_1 \notin (\Lambda_2 \cap \Lambda_1) + (\Lambda_2 \cap \Lambda_3)$. We show it by using contradiction. So let us assume that $\pi_E^7 E_2 + E_1 \in (\Lambda_2 \cap \Lambda_1) + (\Lambda_2 \cap \Lambda_3)$. This means that there is $a \in (\Lambda_2 \cap \Lambda_1)$ and $b \in (\Lambda_2 \cap \Lambda_3)$ such that $\pi_E^7 E_2 + E_1 = a + b$. We have that $a = a_1 E_1 + a_2 E_2$ and $b = b_1 E_1 + b_2 E_2$ for some $a_1, a_2, b_1, b_2 \in E$. Looking at the description of $\Lambda_1$, $\Lambda_2$, $\Lambda_3$, we have that $val_E(a_2) \geq max\{5, 7\} = 7$ and $val_E(b_2) \geq max\{7, 9\} = 9$. This means that we must have $val_E(a_2) = 7$ which having in mind that $a \in \Lambda_1$, gives that $a = \pi_E^2 o_1^\times(\pi_E^5 E_2 + E_1) + o_2 \pi_E^4 E_1 = \pi_E^7 o_1^\times E_2 + (o_2 \pi_E^4 + o_1^\times \pi_E^2)E_1$ for some $o_1^\times \in \mathcal{O}_E^\times$ and some $o_2 \in \mathcal{O}_E$. This implies that we must have $val_E(b_1) = 0$. So since $b \in \Lambda_2$ we have that $b = o_3(\pi_E^2 E_1) + o_4^\times(\pi_E^7 E_2 + E_1)$, where $o_3 \in \mathcal{O}_E$ and $o_4^\times \in \mathcal{O}_E^\times$. Since we also have that $b \in \Lambda_3$ this implies that $b = o_5 E_1 + o_6(\pi_E^9 E_2 + E_1)$, where $o_5, o_6 \in \mathcal{O}_E$. Comparing different expressions for $b$ and coefficients in front of $E_2$ we have $o_4^\times \pi_E^7 = o_6 \pi_E^9$, which looking at valuations on the left and right hand side is not possible. This finishes the proof. $\square$

**Lemma 2.3.2.** *Let $\rho : G \to E$ be a representation and let $L$ be an extension of $E$. If the family of all stable lattices in $\rho$ is distributive it does not imply that the family of all stable lattices in $\rho \otimes_E L$ is distributive as well.*

*Proof.* It follows from lemma 2.3.1 and the fact that for $E = \mathbb{Q}_2$ the lattices have the distributive property. $\square$

### 2.3.2 An induced representation of the mod $p$ Heisenberg group

In this case we look at lattices in irreducible $p$-adic representations of the mod $p$ Heisenberg group, that we denote by $\mathcal{H}_p$. This group can be defined as

$$\mathcal{H}_p := \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & d \\ 0 & 0 & 1 \end{pmatrix} : a, b, d \in \mathbb{F}_p \right\}. \tag{2.3.7}$$

Let us denote generators of the group $g_1 := \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $g_2 := \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $g_3 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Notice that $\mathcal{H}_p$ is non-abelian and its $p$-Sylow subgroup is the whole group. Hence its $p$-Sylow subgroup is normal and non-abelian. Using theorem 1.1.2 we have that for any lattice of a $p$-adic representation, the lattice modulo the field uniformizer has repeated Jordan Holder factors which are trivial. Let us denote by $A$ and $B$ the subgroups of $\mathcal{H}_p$ defined by

$$A := \{g_1^a : a \in \{0, ..., p-1\}\}$$

and

$$B := \left\{ g_2^b : b \in \{0, ..., p-1\} \right\}.$$

We have that $AB \cong \mathbb{F}_p \oplus \mathbb{F}_p$. Let $E := \mathbb{Q}_p(\zeta_p)$, where $\zeta_p$ is a primitive $p$-th root of unity. Define the map

$$\chi_{AB} : AB \to E$$

$$g_1 \mapsto 1$$

$$g_2 \mapsto \zeta_p^c$$

where $c \in \{0, ..., p-1\}$ and $c \neq 0$. Using proposition 9 in [20] we have that the representation

$$\rho := ind_{AB}^{\mathcal{H}_p} \chi_{AB}$$

is irreducible. The representation space of $\rho$ is the family of functions $f : \mathcal{H}_p \to E$ such that for any $a \in AB$ and $h \in \mathcal{H}_p$ we have that $f(ah) = \chi_{AB}(a) f(h)$ and the group $\mathcal{H}_p$ acts on this space by right translation, that is for any $h \in \mathcal{H}_p$ and $f \in \rho$ we have $(hf)(x) = f(xh)$.

We proceed by finding all the stable $\mathcal{O}_E[\mathcal{H}_p]$ lattices inside $\rho$. Let us denote by $\phi_{AB} \in ind_{AB}^{\mathcal{H}_p} \chi_{AB}$ the function which is supported on $AB$ and for which we have $g_1 \mapsto 1$ and $g_2 \mapsto 1$. Define $F_x := g_3^x \phi_{AB}$. We have that $\{F_x : x \in \{0, ..., p-1\}\}$ makes a basis of $ind_{AB}^{\mathcal{H}_p} \chi_{AB}$, which we denote by $\mathcal{B}_{\mathcal{H}}$. Let us compute the action of the generators of $\mathcal{H}_p$ given by $g_1$, $g_2$ and $g_3$ with respect to $\mathcal{B}_{\mathcal{H}}$. We have

$$\begin{aligned}
\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} F_x(y) &= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} x\phi_{AB}(y) \\
&= \phi_{AB}\left( y \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} x \right) \\
&= \phi_{AB}\left( \left( \begin{pmatrix} 1 & a & b \\ 0 & 1 & d \\ 0 & 0 & 1 \end{pmatrix} \right) \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} x \right) \\
&= \phi_{AB}\left( \left( \begin{pmatrix} 1 & a+1 & b \\ 0 & 1 & d \\ 0 & 0 & 1 \end{pmatrix} \right) x \right) \\
&= \phi_{AB}\left( \left( \begin{pmatrix} 1 & 1 & -d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \begin{pmatrix} 1 & a & b \\ 0 & 1 & d \\ 0 & 0 & 1 \end{pmatrix} x \right) \\
&= \chi_{AB}\left( \left( \begin{pmatrix} 1 & 1 & -d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \right) \phi_{AB}\left( \left( \begin{pmatrix} 1 & a & b \\ 0 & 1 & d \\ 0 & 0 & 1 \end{pmatrix} \right) x \right) \\
&= \chi_{AB}\left( \left( \begin{pmatrix} 1 & 1 & -d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \right) F_x(y) \\
&= \chi_{AB}\left( \left( \begin{pmatrix} 1 & 1 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \right) F_x(y),
\end{aligned} \tag{2.3.8}$$

with the last line being true since $F_x$ is supported on $ABx^{-1} = \begin{pmatrix} 1 & a & -ax+b \\ 0 & 1 & -x \\ 0 & 0 & 1 \end{pmatrix}$.

We now compute the action of $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$:

$$\begin{aligned}
\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} F_x(y) &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} x\phi_{AB}(y) \\
&= \phi_{AB}\left( y \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} x \right) \\
&= \phi_{AB}\left( \begin{pmatrix} 1 & a & b \\ 0 & 1 & d \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} x \right) \\
&= \phi_{AB}\left( \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a & b \\ 0 & 1 & d \\ 0 & 0 & 1 \end{pmatrix} x \right) \\
&= \chi_{AB}\left( \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) F_x(y).
\end{aligned} \tag{2.3.9}$$

Finally we compute the action of $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$:

$$\begin{aligned}
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} F_x(y) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} x\phi_{AB}(y) \\
&= \phi_{AB}\left( y \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} x \right) \\
&= \phi_{AB}\left( y \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \left( \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \right)\right) \\
&= \phi_{AB}\left( y \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x+1 \\ 0 & 0 & 1 \end{pmatrix} \right) \\
&= F_{x+1}(y).
\end{aligned} \tag{2.3.10}$$

Using 2.3.8, 2.3.9 and 2.3.10 we compute all the lattices in $ind_{AB}^{\mathcal{H}}\chi_{AB}$ over $\mathbb{Q}_p(\zeta_p)$. We use the Magma code described in section 4. The graph for the lattices for the irreducible representation over $\mathbb{Q}_3(\zeta_3)$ is shown in the figure 2.2. It is interesting to note that the graph is actually four triangles meeting along an edge. Using theorem 2.2.7 we have that any cycle for this graph is of length a multiple of 3 and we can see that all the cycles are triangles with common edge $2 \to 3$. We compute the graph for $p = 5$ as well. The number of non-homothetic lattices in this case is 280, with 2 nodes of degree 39, 18 nodes of degree 14, 110 nodes of degree 9 and 150 nodes of degree 4. Theorem 2.2.7 also tells us that all the cycles for this graph are multiples of 5. Also using the code shown in 4.3.3 we see that the set of stable lattices is distributive for $p = 2$ and not distributive for $p = 3$ and $p = 5$.

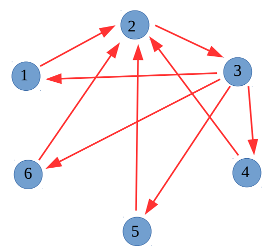


**Figure 2.2:** Heisenberg group

### 2.3.3 A 3-dimensional irreducible representation of $A_4$

In this case we study lattices in the 3-dimensional irreducible representation of the alternating group $A_4$. $A_4$ is the subgroup of even permutations of the symmetric group $S_4$. It is a non-abelian group with abelian and normal 2-Sylow subgroup isomorphic to the Klein four group and with abelian 3-Sylow subgroup isomorphic to the cyclic group of 3 elements. The elements $g_1 = (12)(34)$ and $g_2 = (123)$ are generating $A_4$. We look into the 3-dimensional irreducible representation of $A_4$ over $\mathbb{Q}_2$. It is constructed in the following way. We consider the representation on the 4-dimensional vector space on which each element from $A_4$ acts via applying the permutation on the vector. This representation has for a subrepresentation the subspace of vectors of which the sum of all coefficients is equal to 0. This subrepresentation is irreducible and of dimension 3. Its basis is made of vectors $e_1 = (1, -1, 0, 0)$, $e_2 = (1, 0, -1, 0)$ and $e_3 = (1, 0, 0, -1)$. We compute that $g_1 \cdot e_1 = -e_1$, $g_1 \cdot e_2 = -e_1 + e_3$ and $g_1 \cdot e_3 = -e_1 + e_2$. We also compute that $g_2 \cdot e_1 = -e_1 + e_2$, $g_2 \cdot e_2 = -e_1$ and $g_2 \cdot e_3 = -e_1 + e_3$.

We consider representations over $\mathbb{Q}_2$ and $\mathbb{Q}_3$. Using this we compute all the lattices as described in section 4. We have that when looking over $\mathbb{Q}_2$, the Jordan Holder factors modulo 2 are distinct of dimension 2 and 1. The graph is shown in the figure 2.3.

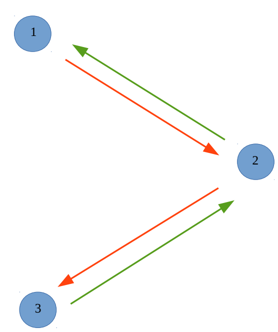


**Figure 2.3:** A4: 3-dimensional representation over $\mathbb{Q}_2$

When we consider the same representation over $\mathbb{Q}_2(\zeta_3)$. In this case, the Jordan Holder factors are all distinct of dimension 1. The graph is shown in the figure 2.4.

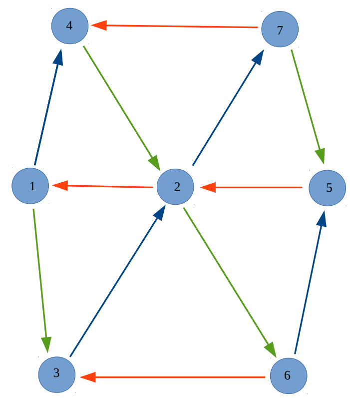


**Figure 2.4:** A4: 3-dimensional representation over $\mathbb{Q}_2(\zeta_3)$

On the other hand, when considering over $\mathbb{Q}_3$, the representation stays irreducible. Using the code shown in 4.3.3 we have that distributivity property is satisfied for both representations above.

### 2.3.4 The 3-dimensional irreducible representation of $S_4$

In this case we consider the 3-dimensional irreducible representation of the symmetric group on 4 elements. The representation which we are looking at is the 3-dimensional irreducible representation over the $p$-adics, obtained in the following way. We look at the natural action of $S_4$ on the 4-dimensional vector space $V$. This representation is not irreducible and has a stable subspace consisting of the vectors whose coefficients sum to 0. So if $e_1, e_2, e_3, e_4$ is a basis of $V$, we have that $e_1 - e_4, e_2 - e_4, e_3 - e_4$ is a basis of the 3-dimensional subrepresentation. This representation is irreducible and it is called the standard representation, which we denote by $\rho$.

We look at the representation over $\mathbb{Q}_2$. We have that $\rho$ is reducible modulo 2, with Jordan-Holder factors of dimension 1 and 2. We compute its lattices using the Magma code 4.3 and its graph is shown in the figure 2.5. For the computaton in Magma, we used that $(1,2),(1,2,3,4) \in S_4$ generate $S_4$. Also, using the code shown in 4.3.3 we have that distributivity property is satisfied.

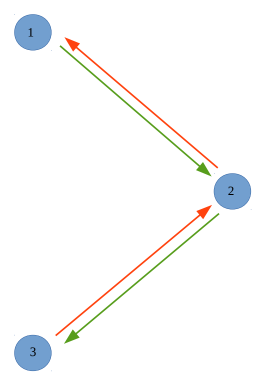


**Figure 2.5:** Symmetric group $S_4$

### 2.3.5 The 5-dimensional irreducible representation of the group $A_5$

In this case we study the lattices in the 5-dimensional irreducible representation of the alternating group $A_5$. Its 2-Sylow subgroup is abelian and isomorphic to the Klein four group, while its 3 and 5 Sylow subgroups are cyclic. We construct the representation by looking at the action of $A_5$ via conjugation on the subgroups of $A_5$ of order 5. We have that the subgroups of order 5 are

$$E_1 := \{(), (1,2,3,5,4), (1,3,4,2,5), (1,5,2,4,3), (1,4,5,3,2)\},$$

$$E_2 := \{(), (1,2,4,3,5), (1,4,5,2,3), (1,3,2,5,4), (1,5,3,4,2)\},$$

$$E_3 := \{(), (1,2,4,5,3), (1,4,3,2,5), (1,5,2,3,4), (1,3,5,4,2)\},$$

$$E_4 := \{(), (1,2,5,3,4), (1,5,4,2,3), (1,3,2,4,5), (1,4,3,5,2)\},$$

$$E_5 := \{(), (1,2,5,4,3), (1,5,3,2,4), (1,4,2,3,5), (1,3,4,5,2)\},$$

$$E_6 := \{(), (1,2,3,4,5), (1,3,5,2,4), (1,4,2,5,3), (1,5,4,3,2)\}.$$

We first compute the action of $A_5$ on the 6-dimensional vector space $V$ over the $p$-adics with basis vectors $E_1, E_2, E_3, E_4, E_5, E_6$. We have that $g_1 := (1,2,3,4,5)$ and $g_2 := (1,2,3)$ generate $A_5$. We compute that $g_1 \cdot (1,2,3,5,4) = (1,5,2,3,4)$, $g_1 \cdot (1,2,4,3,5) = (1,2,3,5,4)$, $g_1 \cdot (1,2,4,5,3) = (1,4,2,3,5)$, $g_1 \cdot (1,2,5,3,4) = (1,4,5,2,3)$, $g_1 \cdot (1,2,5,4,3) = (1,5,4,2,3)$, $g_1 \cdot (1,2,3,4,5) = (1,2,3,4,5)$. This gives that $g_1$ acts on the 6-dimensional vector space $V$ via the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We also compute that $g_2 \cdot (1,2,3,5,4) = (1,5,4,2,3)$, $g_2 \cdot (1,2,4,3,5) = (1,4,3,2,5)$, $g_2 \cdot (1,2,4,5,3) = (1,2,3,4,5)$, $g_2 \cdot (1,2,5,3,4) = (1,4,2,3,5)$, $g_2 \cdot (1,2,5,4,3) = (1,2,3,5,4)$, $g_2 \cdot (1,2,3,4,5) = (1,4,5,2,3)$. This gives that $g_2$ acts on the $6-$dimensional vector space $V$ via the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

This representation is not irreducible. It has a stable 1-dimensional subspace with basis vector $E_1 + E_2 + E_3 + E_4 + E_5 + E_6$. The 5-dimensional quotient is irreducible. The action of $g_1$ and $g_2$ on the 5-dimensional vector space is via the matrices

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{pmatrix}$$

respectively, where the action is given with respect to the basis which is image of $E_1, E_2, E_3, E_4, E_5$. Using this we compute in Magma all the lattices over $\mathbb{Q}_2(\zeta_3)$. We have that the Jordan-Holder factors of the lattices modulo 2 are non isomorphic of dimensions 2, 2 and 1. The graph is shown in the figure 2.6. Also, using the code shown in 4.3.3 we have that distributivity property is satisfied.

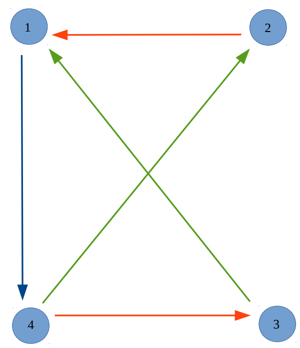


**Figure 2.6:** A5: 5 dimensional representation over $\mathbb{Q}_2(\zeta_3)$

### 2.3.6 A 5-dimensional irreducible representation of the group $S_5$

In this case we study lattices in a 5-dimensional irreducible representation of the symmetric group $S_5$. Its 2-Sylow subgroup is non-abelian and isomorphic to the dihedral group $D_8$. Its 3 and 5 Sylow subgroups are cyclic. We construct the representation using Young tableaux, as described in [15]. The 5 dimensional irreducible representation which we are studying is the one described by the tableaux of shape

$$\begin{array}{|c|c|c|}\hline \phantom{1} & \phantom{1} & \phantom{1} \\ \hline \phantom{1} & \phantom{1} \\ \cline{1-2}\end{array}.$$

We consider Young tableaux up to row permutations, that is

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array} \sim \begin{array}{|c|c|c|}\hline 1 & 4 & 3 \\ \hline 5 & 2 \\ \cline{1-2}\end{array}.$$

The representatives of non-equivalent Young tableaux of this shape are

$$T_1 := \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array} \qquad T_2 := \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array} \qquad T_3 := \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}$$

$$T_4 := \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array} \qquad T_5 := \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}$$

With $\{T\}$ we denote the class of tableaux equivalent to $T$ and with $C(T)$ we denote the column stabilizer of $T$. Using this, we have that the following vectors make a basis of the irreducible $5-$ dimensional representation:

$$\begin{aligned}
e_{T_1} &:= \sum_{P \in C(T_1)} sgn(P)P(T_1) \\
&= \left\{\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}\right\} + \left\{\begin{array}{|c|c|c|}\hline 4 & 5 & 3 \\ \hline 1 & 2 \\ \cline{1-2}\end{array}\right\} - \left\{\begin{array}{|c|c|c|}\hline 4 & 2 & 3 \\ \hline 1 & 5 \\ \cline{1-2}\end{array}\right\} - \left\{\begin{array}{|c|c|c|}\hline 1 & 5 & 3 \\ \hline 4 & 2 \\ \cline{1-2}\end{array}\right\} \\
e_{T_2} &:= \sum_{P \in C(T_2)} sgn(P)P(T_2) \\
&= \left\{\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array}\right\} + \left\{\begin{array}{|c|c|c|}\hline 2 & 5 & 4 \\ \hline 1 & 3 \\ \cline{1-2}\end{array}\right\} - \left\{\begin{array}{|c|c|c|}\hline 2 & 3 & 4 \\ \hline 1 & 5 \\ \cline{1-2}\end{array}\right\} - \left\{\begin{array}{|c|c|c|}\hline 1 & 5 & 4 \\ \hline 2 & 3 \\ \cline{1-2}\end{array}\right\} \\
e_{T_3} &:= \sum_{P \in C(T_3)} sgn(P)P(T_3) \\
&= \left\{\begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}\right\} + \left\{\begin{array}{|c|c|c|}\hline 2 & 4 & 5 \\ \hline 1 & 3 \\ \cline{1-2}\end{array}\right\} - \left\{\begin{array}{|c|c|c|}\hline 1 & 4 & 5 \\ \hline 2 & 3 \\ \cline{1-2}\end{array}\right\} - \left\{\begin{array}{|c|c|c|}\hline 2 & 3 & 5 \\ \hline 1 & 4 \\ \cline{1-2}\end{array}\right\} \\
e_{T_4} &:= \sum_{P \in C(T_4)} sgn(P)P(T_4) \\
&= \left\{\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}\right\} + \left\{\begin{array}{|c|c|c|}\hline 3 & 5 & 4 \\ \hline 1 & 2 \\ \cline{1-2}\end{array}\right\} - \left\{\begin{array}{|c|c|c|}\hline 1 & 5 & 4 \\ \hline 3 & 2 \\ \cline{1-2}\end{array}\right\} - \left\{\begin{array}{|c|c|c|}\hline 3 & 2 & 4 \\ \hline 1 & 5 \\ \cline{1-2}\end{array}\right\} \\
e_{T_5} &:= \sum_{P \in C(T_5)} sgn(P)P(T_5) \\
&= \left\{\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}\right\} + \left\{\begin{array}{|c|c|c|}\hline 3 & 4 & 5 \\ \hline 1 & 2 \\ \cline{1-2}\end{array}\right\} - \left\{\begin{array}{|c|c|c|}\hline 1 & 4 & 5 \\ \hline 3 & 2 \\ \cline{1-2}\end{array}\right\} - \left\{\begin{array}{|c|c|c|}\hline 3 & 2 & 5 \\ \hline 1 & 4 \\ \cline{1-2}\end{array}\right\}
\end{aligned}$$

We have that $g_1 := (1,2)(3,4)$, $g_2 := (2,3)(4,5)$ and $g_3 := (1,2)$ generate $S_5$. We study their action on this 5-dimensional representation. We have that

$$
\begin{aligned}
g_1 e_{T_1} &= \left\{\begin{ytableau} 2 & 1 & 4 \\ 3 & 5 \end{ytableau}\right\} + \left\{\begin{ytableau} 3 & 5 & 4 \\ 2 & 1 \end{ytableau}\right\} - \left\{\begin{ytableau} 3 & 1 & 4 \\ 2 & 5 \end{ytableau}\right\} - \left\{\begin{ytableau} 2 & 5 & 4 \\ 3 & 1 \end{ytableau}\right\} \\
&= e_{T_4} - e_{T_2} \\
g_1 e_{T_2} &= \left\{\begin{ytableau} 2 & 4 & 3 \\ 1 & 5 \end{ytableau}\right\} + \left\{\begin{ytableau} 1 & 5 & 3 \\ 2 & 4 \end{ytableau}\right\} - \left\{\begin{ytableau} 1 & 4 & 3 \\ 2 & 5 \end{ytableau}\right\} - \left\{\begin{ytableau} 2 & 5 & 3 \\ 1 & 4 \end{ytableau}\right\} \\
&= -e_{T_2} + e_{T_3} \\
g_1 e_{T_3} &= \left\{\begin{ytableau} 2 & 4 & 5 \\ 1 & 3 \end{ytableau}\right\} + \left\{\begin{ytableau} 1 & 3 & 5 \\ 2 & 4 \end{ytableau}\right\} - \left\{\begin{ytableau} 2 & 3 & 5 \\ 1 & 4 \end{ytableau}\right\} - \left\{\begin{ytableau} 1 & 4 & 5 \\ 2 & 3 \end{ytableau}\right\} \\
&= e_{T_3} \\
g_1 e_{T_4} &= \left\{\begin{ytableau} 2 & 1 & 3 \\ 4 & 5 \end{ytableau}\right\} + \left\{\begin{ytableau} 4 & 5 & 3 \\ 2 & 1 \end{ytableau}\right\} - \left\{\begin{ytableau} 2 & 5 & 3 \\ 4 & 1 \end{ytableau}\right\} - \left\{\begin{ytableau} 4 & 1 & 3 \\ 2 & 5 \end{ytableau}\right\} \\
&= e_{T_1} - e_{T_2} + e_{T_3} \\
g_1 e_{T_5} &= \left\{\begin{ytableau} 2 & 1 & 5 \\ 4 & 3 \end{ytableau}\right\} + \left\{\begin{ytableau} 4 & 3 & 5 \\ 2 & 1 \end{ytableau}\right\} - \left\{\begin{ytableau} 2 & 3 & 5 \\ 4 & 1 \end{ytableau}\right\} - \left\{\begin{ytableau} 4 & 1 & 5 \\ 2 & 3 \end{ytableau}\right\} \\
&= e_{T_5}
\end{aligned}
$$

This gives us that $g_1$ acts via the matrix

$$
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
-1 & -1 & 0 & -1 & 0 \\
0 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.
$$

Similarly, we compute that $g_2$ acts via

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

and $g_3$ acts via

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Using this we compute lattices in irreducible representations over $\mathbb{Q}_2$, $\mathbb{Q}_3$ and $\mathbb{Q}_5$. We have that the Jordan Holder factors modulo 2 and 3 are of dimension 1 and 4, but modulo 5 it stays irreducible. Also notice that comparing the character of this representation and the character of the representation studied in 2.3.5 we see that the 5- dimensional irreducible representation of $A_5$ is the restriction of this 5- dimensional irreducible representation of $S_5$. The corresponding graphs of lattices in irreducible representations over $\mathbb{Q}_2$, $\mathbb{Q}_3$ are isomorphic and they are shown in the figure 2.7. Just looking at the graph or using the code shown in 4.3.3 we see that distributivity property is satisfied for both representations.


**Figure 2.7:** S5: 5 dimensional representation over $\mathbb{Q}_2$, $\mathbb{Q}_3$

### 2.3.7 The 6-dimensional irreducible representation of the group $S_5$

In this case we study the 6-dimensional irreducible representation of the symmetric group $S_5$. The construction will be done using Young tableaux, as in 2.3.6. The tableau shape that generates this representation is

$$\begin{array}{|c|c|c|}\hline \phantom{1} & \phantom{1} & \phantom{1} \\ \hline \phantom{1} & \multicolumn{2}{c}{} \\ \cline{1-1} \phantom{1} & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}.$$

The set of non-equivalent tableaux is

$$T_1 := \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & \multicolumn{2}{c}{} \\ \cline{1-1} 5 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \qquad T_2 := \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & \multicolumn{2}{c}{} \\ \cline{1-1} 5 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \qquad T_3 := \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & \multicolumn{2}{c}{} \\ \cline{1-1} 4 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

$$T_4 := \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1} 5 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \qquad T_5 := \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1} 4 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \qquad T_6 := \begin{array}{|c|c|c|}\hline 1 & 4 & 5 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1} 3 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

They lead to the basis

$$
\begin{aligned}
e_{T_1} &:= \sum_{P\in C(T_1)} sgn(P)P(T_1) \\
&= \left\{\begin{ytableau} 4 & 2 & 3 \\ 5 \\ 1 \end{ytableau}\right\} + \left\{\begin{ytableau} 5 & 2 & 3 \\ 1 \\ 4 \end{ytableau}\right\} + \left\{\begin{ytableau} 1 & 2 & 3 \\ 4 \\ 5 \end{ytableau}\right\} \\
&- \left\{\begin{ytableau} 4 & 2 & 3 \\ 1 \\ 5 \end{ytableau}\right\} - \left\{\begin{ytableau} 5 & 2 & 3 \\ 4 \\ 1 \end{ytableau}\right\} - \left\{\begin{ytableau} 1 & 2 & 3 \\ 5 \\ 4 \end{ytableau}\right\} \\
e_{T_2} &:= \sum_{P\in C(T_2)} sgn(P)P(T_2) \\
&= \left\{\begin{ytableau} 1 & 2 & 4 \\ 3 \\ 5 \end{ytableau}\right\} + \left\{\begin{ytableau} 3 & 2 & 4 \\ 5 \\ 1 \end{ytableau}\right\} + \left\{\begin{ytableau} 5 & 2 & 4 \\ 1 \\ 3 \end{ytableau}\right\} \\
&- \left\{\begin{ytableau} 3 & 2 & 4 \\ 1 \\ 5 \end{ytableau}\right\} - \left\{\begin{ytableau} 5 & 2 & 4 \\ 3 \\ 1 \end{ytableau}\right\} - \left\{\begin{ytableau} 1 & 2 & 4 \\ 5 \\ 3 \end{ytableau}\right\} \\
e_{T_3} &:= \sum_{P\in C(T_3)} sgn(P)P(T_3) \\
&= \left\{\begin{ytableau} 1 & 2 & 5 \\ 3 \\ 4 \end{ytableau}\right\} + \left\{\begin{ytableau} 3 & 2 & 5 \\ 4 \\ 1 \end{ytableau}\right\} + \left\{\begin{ytableau} 4 & 2 & 5 \\ 1 \\ 3 \end{ytableau}\right\} \\
&- \left\{\begin{ytableau} 3 & 2 & 5 \\ 1 \\ 4 \end{ytableau}\right\} - \left\{\begin{ytableau} 4 & 2 & 5 \\ 3 \\ 1 \end{ytableau}\right\} - \left\{\begin{ytableau} 1 & 2 & 5 \\ 4 \\ 3 \end{ytableau}\right\} \\
e_{T_4} &:= \sum_{P\in C(T_4)} sgn(P)P(T_4) \\
&= \left\{\begin{ytableau} 1 & 3 & 4 \\ 2 \\ 5 \end{ytableau}\right\} + \left\{\begin{ytableau} 2 & 3 & 4 \\ 5 \\ 1 \end{ytableau}\right\} + \left\{\begin{ytableau} 5 & 3 & 4 \\ 1 \\ 2 \end{ytableau}\right\} \\
&- \left\{\begin{ytableau} 2 & 3 & 4 \\ 1 \\ 5 \end{ytableau}\right\} - \left\{\begin{ytableau} 5 & 3 & 4 \\ 2 \\ 1 \end{ytableau}\right\} - \left\{\begin{ytableau} 1 & 3 & 4 \\ 5 \\ 2 \end{ytableau}\right\}
\end{aligned}
$$

$$
\begin{aligned}
e_{T_5} &:= \sum_{P \in C(T_5)} sgn(P)P(T_5) \\
&= \left\{ \begin{ytableau} 1 & 3 & 5 \\ 2 \\ 4 \end{ytableau} \right\} + \left\{ \begin{ytableau} 2 & 3 & 5 \\ 4 \\ 1 \end{ytableau} \right\} + \left\{ \begin{ytableau} 4 & 3 & 5 \\ 1 \\ 2 \end{ytableau} \right\} \\
&- \left\{ \begin{ytableau} 2 & 3 & 5 \\ 1 \\ 4 \end{ytableau} \right\} - \left\{ \begin{ytableau} 4 & 3 & 5 \\ 2 \\ 1 \end{ytableau} \right\} - \left\{ \begin{ytableau} 1 & 3 & 5 \\ 4 \\ 2 \end{ytableau} \right\} \\
e_{T_6} &:= \sum_{P \in C(T_6)} sgn(P)P(T_5) \\
&= \left\{ \begin{ytableau} 1 & 4 & 5 \\ 2 \\ 3 \end{ytableau} \right\} + \left\{ \begin{ytableau} 2 & 4 & 5 \\ 3 \\ 1 \end{ytableau} \right\} + \left\{ \begin{ytableau} 3 & 4 & 5 \\ 1 \\ 2 \end{ytableau} \right\} \\
&- \left\{ \begin{ytableau} 2 & 4 & 5 \\ 1 \\ 3 \end{ytableau} \right\} - \left\{ \begin{ytableau} 3 & 4 & 5 \\ 2 \\ 1 \end{ytableau} \right\} - \left\{ \begin{ytableau} 1 & 4 & 5 \\ 3 \\ 2 \end{ytableau} \right\}
\end{aligned}
$$

As in the previous case, we compute that $g_1$ acts via

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
-1 & -1 & 0 & -1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & -1 \\
1 & 0 & -1 & 0 & -1 & 0
\end{pmatrix},
$$

$g_2$ acts via

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

and $g_3$ acts via

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \end{pmatrix}.$$

Using this we compute in Magma lattices in irreducible representations over $\mathbb{Q}_2$, $\mathbb{Q}_3$ and $\mathbb{Q}_5$. We have that the Jordan-Holder factors modulo 2 are repeated of dimension 1 and 4, modulo 3 it stays irreducible, while modulo 5 they are non-isomorphic of dimension 3. The graph corresponding to the representation over $\mathbb{Q}_2$ is shown in the figure 2.8

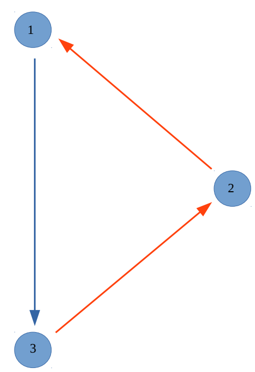


**Figure 2.8:** S5: 6 dimensional representation over $\mathbb{Q}_2$

and the graph corresponding to the representation over $\mathbb{Q}_5$ is shown in the figure 2.9.


**Figure 2.9:** S5: 6 dimensional representation over $\mathbb{Q}_5$

Just looking at the graphs or using the code shown in 4.3.3 we have that distributivity property is satisfied for both representations above.

### 2.3.8 Graphs of absolutely irreducible representations with abelian $p$-Sylow subgroup

Let $G$ be a finite group with an abelian $p$-Sylow subgoup. We study if for an absolutely irreducible representation $\rho$ of $G$ over the $p$-adics the associated set of lattices has the distributive property. Using theorem 2.1.6 we get that this is equivalent to the reduction being multiplicity-free if $\Lambda_\rho$ is an intersection of maximal orders, where $\Lambda_\rho$ is as in the beginning of the section.

For a positive integer $x$ we define $x_p := p^d$ where $d$ is the largest integer such that $p^d \mid x$. For a representation $\rho$ of a group $G$, we define the defect to be the integer $d_\rho$ such that $p^{d_\rho}|dim(\rho)|_p = |G|_p$. Using Brauer-Nesbitt theorem on blocks of defect zero stated in [2] we have that irreducible characters of defect 0 reduce to irreducible modular characters. In [1] Brauer showed that all characters of defect one reduce to a sum of irreducible modular characters with decomposition numbers less or equal to one. This gives us that a representation of defect 0 or 1 has no repeated Jordan-Holder factors. So this implies that we need to look at groups whose $p$-Sylow subgroup has order at least $p^2$. The first groups that we will look into are of order $2^a 3^b$ where at least one of $a$ or $b$ is larger than 1. We check if an absolutely irreducible representation has repeated Jordan Holder factors using Brauer theory, as described in 1.3.

In 4.3.2 is the Magma code which for a given integer `order` checks if there is an absolutely irreducible representation of a group of order `order` with the wanted properties. The code for the output has an array for which each element is an array of type `[n, p, i]`. This means that the group `SmallGroup(order,n)` has an abelian $p$-Sylow subgroup. Moreover there is an absolutely irreducible representation of $G$

with character $CharacterTable(G)[i]$ with repeated Jordan-Holder factors modulo uniformiser over the field that the representation is realised (an extension of $\mathbb{Q}_p$). We check this by expressing the character of the representation restricted to the subgroup of $p-$regular elements of $G$ as a linear combination of Brauer characters. This decomposition can be read in the code from the matrix `decompo`. We saw that for `order` $= 72$ there are 4 cases with the wanted properties. All of them are for $p = 3$, but for different groups. These groups are `SmallGroup(72, 22)`, `SmallGroup(72, 23)`, `SmallGroup(72, 24)` and `SmallGroup(72, 41)`. In the following cases, we study their representations with repeated Jordan-Holder factors modulo a uniformiser over the field that the representation is realised (an extension of $\mathbb{Q}_3$).

We first study the absolutely irreducible representation of $G :=$ `SmallGroup(72, 22)` with repeated Jordan-Holder factors. Looking at the value of the character of the representation at 1 we see that the dimension of the representaton is 4. Representing the character of the representation restricted to the subgroup of $G$ of $3-$ regular elements we see that the set of Jordan-Holder factors is made up of two isomorphic quotients of dimension 2. By criteria that we used in order to find the repesentation we have that the 3-Sylow subgroup of `SmallGroup(72, 22)` is abelian and it is of order 9. We check that the 3-Sylow subgroup is normal and isomorphic to the direct product of cyclic groups $C_3$ of order 3. Also its 2-Sylow subgroup is non-abelian, non-normal in $G$ and isomorphic to the dihedral group $D_8$ of order 8. The fact that 3- Sylow is normal gives us that Jordan Holder factors of the reduction are representations of the quotient $D_8$. Since 2- Sylow is not normal we have that `SmallGroup(72, 22)` is not isomorphic to a direct product of its 3-Sylow and 2-Sylow subgroups, but since 3-Sylow is normal we have that `SmallGroup(72, 22)` is isomorphic to a semidirect product of its 3-Sylow and 2-Sylow subgroups. We have that $G = \langle \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5 \mid \sigma_1^2 = id, \sigma_2^2 = id, \sigma_3^2 = id, \sigma_4^3 = id, \sigma_5^3 = id, \sigma_1^{-1}\sigma_2\sigma_1 = \sigma_2\sigma_3, \sigma_2^{-1}\sigma_4\sigma_2 = \sigma_4^2, \sigma_1^{-1}\sigma_5\sigma_1 = \sigma_5^2 \rangle$.

We consider the representation over $\mathbb{Q}_3(\zeta_3)$. Using Magma we compute the action of generators of the group $G$. For example, we know that the generator $\sigma_1 = G.1$ acts as `Representation(IrreducibleModules(G,CyclotomicField(3))[15])(G.1)`, since we know that the corresponding output from the Magma code is `[72, 3, 15]`. The action of the generators is given by

$$\sigma_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\sigma_4 = \begin{pmatrix} \zeta_3 & 0 & 0 & 0 \\ 0 & \zeta_3^2 & 0 & 0 \\ 0 & 0 & \zeta_3 & 0 \\ 0 & 0 & 0 & \zeta_3^2 \end{pmatrix}, \sigma_5 = \begin{pmatrix} \zeta_3 & 0 & 0 & 0 \\ 0 & \zeta_3 & 0 & 0 \\ 0 & 0 & \zeta_3^2 & 0 \\ 0 & 0 & 0 & \zeta_3^2 \end{pmatrix}.$$

The graph is shown in the figure 2.10.

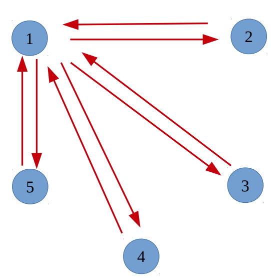

**Figure 2.10:** SmallGroup(72,22)

Let us denote by $\Lambda_1$, $\Lambda_2$, $\Lambda_3$, $\Lambda_4$, $\Lambda_5$ a set of lattices such that the lattice $\Lambda_i$ represents node $i$ for $1 \leq i \leq 5$ and such that $\pi_E \Lambda_1 \subset \Lambda_j \subset \Lambda_1$ and $\Lambda_1/\Lambda_j$ is irreducible for $2 \leq j \leq 5$, which we see is possible after looking at the graph 2.10. Since there is an edge from $\Lambda_j$ to $\Lambda_1$ for $2 \leq j \leq 5$, we have that $\Lambda_j/\pi_E\Lambda_1$ is irreducible. Let us compute $\Lambda_i + \Lambda_j$ and $\Lambda_i \cap \Lambda_j$ where $2 \leq i, j \leq 5$. Notice that we can not have $\Lambda_i \subset \Lambda_j$ since this would mean that $\pi_E\Lambda_1 \subset \Lambda_i \subset \Lambda_j$, and since $\Lambda_i \neq \Lambda_j$ this would imply that that $\Lambda_j/\pi_E\Lambda_1$ is not irreducible. This implies that $\Lambda_j \subsetneq \Lambda_i + \Lambda_j$. We have that $\Lambda_j \subsetneq \Lambda_i + \Lambda_j \subset \Lambda_1$, which having in mind that $\Lambda_1/\Lambda_j$ is irreducible implies that $\Lambda_i + \Lambda_j = \Lambda_1$. On the other side, we have that $\pi_E\Lambda_1 \subset \Lambda_i \cap \Lambda_j \subsetneq \Lambda_i$, which since $\Lambda_i/\Lambda_i \cap \Lambda_j$ implies that $\Lambda_i \cap \Lambda_j = \pi_E\Lambda_1$. Using just shown we have that

$$(\Lambda_2 \cap \Lambda_3) + \Lambda_4 = \pi_E\Lambda_1 + \Lambda_4 = \Lambda_4$$

and

$$(\Lambda_2 + \Lambda_4) \cap (\Lambda_3 + \Lambda_4) = \Lambda_1 \cap \Lambda_1 = \Lambda_1$$

which shows that the family of lattices is not distributive.

We then study the irreducible representation of $G :=$ `SmallGroup(72, 23)`, with character $CharacterTable(G)[15]$. Looking at the character we see that dimension of the representation is 4. We also know that this representation will have repeated Jordan-Holder factors modulo the uniformiser of the extension of $\mathbb{Q}_3$ over which the representation is realised. Representing the character of the representation restricted to the subgroup of $G$ of 3-regular elements in terms of Brauer characters, we see that the set of Jordan-Holder quotients will be made of 2 isomorphic quotients each of dimension 2. Also, we know that the 3-Sylow subgroup of $G$ is abelian. Moreover we check that its 3-Sylow subgroup is normal and isomorphic to the direct product of two cyclic groups $C_3$ of order 3. The 2-Sylow subgroup of $G$ is non-abelian, non-normal and isomorphic to the dihedral group. As before, having in mind that the 3-Sylow subgroup is normal and that the 2-Sylow subgroup is not we conclude that $G$ is isomorphic to a semidirect product of its 2-Sylow and 3-Sylow subgroups. We have that $G = \langle \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5 \mid \sigma_1^2 = id, \sigma_2^2 = \sigma_3, \sigma_3^2 = id, \sigma_4^3 = id, \sigma_5^3 = id, \sigma_1^{-1}\sigma_2\sigma_1 = \sigma_2\sigma_3, \sigma_2^{-1}\sigma_4\sigma_2 = \sigma_4^2, \sigma_1^{-1}\sigma_5\sigma_1 = \sigma_5^2 \rangle$.

We compute the action of generators of $G$. We consider the representation over $\mathbb{Q}_3$ and the action of generators is as follows:

$$\sigma_1 = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 1 & -1 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 1 & 1 & 1 & -1 \\ -1 & -1 & -1 & 2 \\ -2 & -1 & 0 & -1 \\ -1 & -1 & 0 & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\sigma_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & -1 \end{pmatrix}, \sigma_5 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ -2 & -1 & -1 & 0 \\ -1 & 0 & 0 & -1 \end{pmatrix}.$$

The graph is shown in the figure 2.11.


**Figure 2.11:** SmallGroup(72,23)

We then investigate if the family of lattices as described in the absolutely irreducible representation of $SmallGroup(72, 23)$ has a system of compatible basis, defined in 2.1.3. We compute lattices using the code as in 4.3, just with different action generators and different fields $F$ and $E$. Looking at the values of `Qbasis_L` we extracted what the basis is for the representative lattices, up to homothety, of the family. We have that the latices are $\Lambda_1 = E_1 \oplus E_2 \oplus E_3 \oplus E_4$ and $\Lambda_2 = (2 * 3E_2) \oplus (2 * 3E_1 + 2 * 3E_2) \oplus (E_1 + 2E_3 + E_4) \oplus (E_2 + E_3)$, where $E_1$, $E_2$, $E_3$, $E_4$ is the basis with respect to which we were computing the action matrices in the absolutely irreducible representation of $SmallGroup(72, 23)$. Noticing that $\Lambda_1 = E_2 \oplus (E_1 + E_2) \oplus (E_1 + 2E_3 + E_4) \oplus (E_2 + E_3)$ and that $\Lambda_2 = 2 * 3E_2 \oplus 2 * 3(E_1 + E_2) \oplus (E_1 + 2E_3 + E_4) \oplus (E_2 + E_3)$, we have that in this case there is a system of compatible basis, even though we have repeated Jordan Holder factors. That this family of lattices has a system of compatible basis could be seen just by looking at the graph 2.11 and noticing that it implies that the family of stable lattices is well ordered, that is that for every subset $S = \{\Lambda_1, ..., \Lambda_n\}$ of lattices we have that $\bigcap_{i=1}^{n} \Lambda_i \in S$ and $\bigcup_{i=1}^{n} \Lambda_i \in S$.

For the next step we study the representation of the group $G :=$ `SmallGroup(72, 24)`. We know that it has repeated Jordan-Holder factors modulo the uniformiser of the extension over $\mathbb{Q}_3$ over which it is realised. Representing the character restricted to the subgroup of $G$ of 3-regular elements we get that set of Jordan-Holder factors consists of two isomorphic quotients of dimension 2. Also, looking at the character, we see that the representation is 4-dimensional. We know that its 3-Sylow subgroup is abelian. Additionally we check that its 3-Sylow subgroup is normal and isomorphic to the direct product of cyclic groups $C_3$ of order 3. The 2-Sylow subgroup of $G$ is non-abelian and non-normal in $G$ and isomorphic to the quaternion group. As before, having in mind that the 3-Sylow subgroup is normal and that the 2-Sylow subgroup is not we conclude that $G$ is isomorphic to a semidirect product of its 2-Sylow and 3-Sylow subgroups. We have that $G = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5 \mid \sigma_1^2 = \sigma_3, \sigma_2^2 = \sigma_3, \sigma_3^2 = id, \sigma_4^3 = id, \sigma_5^3 = id, \sigma_1^{-1}\sigma_2\sigma_1 = \sigma_2\sigma_3, \sigma_2^{-1}\sigma_4\sigma_2 = \sigma_4^2, \sigma_1^{-1}\sigma_5\sigma_1 = \sigma_5^2\}$.

We compute the action of generators of $G$. The representation is realised over $\mathbb{Q}_3$ and a set of generators acts via the following matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\sigma_4 = \begin{pmatrix} \zeta_3 & 0 & 0 & 0 \\ 0 & -\zeta_3 - 1 & 0 & 0 \\ 0 & 0 & \zeta_3 & 0 \\ 0 & 0 & 0 & -\zeta_3 - 1 \end{pmatrix}, \sigma_5 = \begin{pmatrix} \zeta_3 & 0 & 0 & 0 \\ 0 & \zeta_3 & 0 & 0 \\ 0 & 0 & -\zeta_3 - 1 & 0 \\ 0 & 0 & 0 & -\zeta_3 - 1 \end{pmatrix}.$$

The graph of stable lattices up to homothety is shown in the figure 2.12.

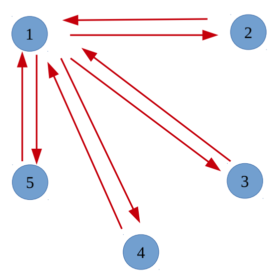


**Figure 2.12:** SmallGroup(72,24)

Looking at the graph we see that it is the same as the graph in the figure 2.10, so since the proof that the family of lattices in the representation of `SmallGroup(72, 22)` is not distributive uses only the graph, the same proof applies in this case too, that is in this case the distributivity property does not hold either.

Lastly, we study the absolutely irreducible representation of $G :=$ `SmallGroup(72, 41)`. Looking at the character of the representation we see that the dimension of the representation is 8. We know that the representation has repeated Jordan-Holder factors and representing the character restricted to the subgroup of $G$ of $p$-regular elements as a linear combination of Brauer characters, we see that the set of Jordan-Holder factors is two repeated quotients of dimension 2 and 4 pairwise non-isomorphic quotients of dimension 1. We also know that the 3-Sylow subgroup of $G$ is abelian. We check that the 3-Sylow subgroup is normal in $G$ and isomorphic to the direct product of cyclic groups $C_3$ of dimension 3. The 2-Sylow subgroup of $G$ is non-abelian and non-normal in $G$ and isomorphic to the quaternion group. As before, having in mind that the 3-Sylow subgroup is normal and that the 2-Sylow subgroup is not we conclude that $G$ is isomorphic to a semidirect product of its 2-Sylow and 3-Sylow subgroups. We have that $G = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5 \mid \sigma_1^2 = \sigma_3, \sigma_2^2 = \sigma_3, \sigma_3^2 = id, \sigma_4^3 = id, \sigma_5^3 = id, \sigma_1^{-1}\sigma_2\sigma_1 = \sigma_2\sigma_3, \sigma_1^{-1}\sigma_4\sigma_1 = \sigma_4\sigma_5^2, \sigma_2^{-1}\sigma_4\sigma_2 = \sigma_5, \sigma_3^{-1}\sigma_4\sigma_3 = \sigma_4^2, \sigma_1^{-1}\sigma_5\sigma_1 = \sigma_4^2\sigma_5^2, \sigma_2^{-1}\sigma_5\sigma_2 = \sigma_4^2, \sigma_3^{-1}\sigma_5\sigma_3 = \sigma_5^2\}$.

The representation is realised over $\mathbb{Q}_3$. We compute the action of a set of generators as in `SmallGroup(72, 22)`. The action of generators represented via matrices is as follows:

$$\sigma_1 = \begin{pmatrix} 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ -2 & 1 & -1 & -1 & 1 & 2 & 0 & -1 \\ 1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ -3 & 1 & -1 & -1 & 1 & 2 & 0 & -1 \\ 2 & 0 & 1 & 1 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & -1 & 0 & 1 & 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 0 & -1 & 0 & 0 & -1 \\ 1 & 1 & 3 & 0 & -3 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 3 & -1 & -3 & -1 & 0 & -2 \\ -1 & 0 & -2 & 1 & 2 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \end{pmatrix},$$

$$\sigma_3 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 3 & 1 & -3 & 0 & 0 & 1 \\ 1 & 0 & 4 & 2 & -3 & 0 & -1 & 2 \\ -1 & 0 & -2 & 0 & 2 & 0 & 0 & -1 \\ 1 & 0 & 5 & 2 & -4 & 0 & -1 & 2 \\ -1 & 0 & -3 & -1 & 3 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 2 & -1 & -2 & 0 & 0 & 0 \end{pmatrix}, \sigma_4 = \begin{pmatrix} -1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 3 & 1 & -2 & 1 & 0 & 0 \\ 1 & 1 & 4 & 1 & -2 & 2 & -1 & 0 \\ -1 & -1 & -2 & -1 & 1 & -1 & 0 & 0 \\ 1 & 2 & 5 & 2 & -3 & 2 & -1 & 0 \\ -1 & -1 & -3 & -1 & 2 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 & -1 & 0 \\ 1 & 1 & 2 & 2 & -1 & 1 & 0 & 1 \end{pmatrix},$$

$$\sigma_5 = \begin{pmatrix} 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 \\ -1 & 1 & -1 & 3 & 2 & 1 & 0 & 1 \\ -1 & 1 & -1 & 4 & 2 & 1 & -1 & 2 \\ 1 & 0 & 1 & -2 & -2 & -1 & 0 & -1 \\ -1 & 1 & -2 & 4 & 3 & 1 & -1 & 2 \\ 1 & -1 & 1 & -3 & -2 & -1 & 1 & -1 \\ 0 & -1 & 0 & 0 & 0 & -1 & -1 & 0 \\ -1 & 0 & -2 & 1 & 2 & 0 & 0 & 0 \end{pmatrix}.$$

The graph of stable lattices is shown in the figure 2.13. Using the code shown in 4.3.3 we have that distributivity property is not satisfied for this representation.

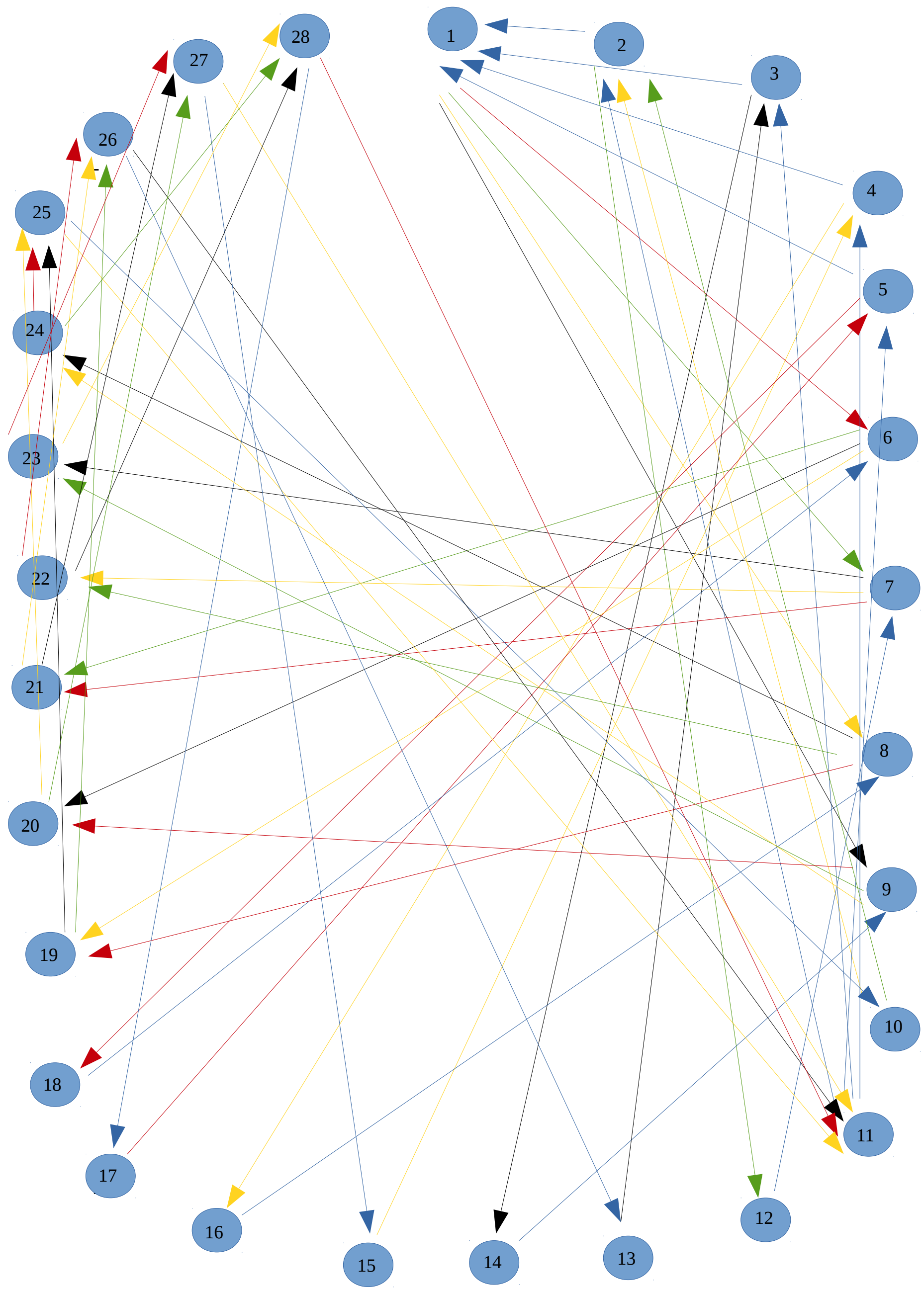


**Figure 2.13:** SmallGroup(72,41)

# Chapter 3

# Description of lattices in principal series representations

## 3.1 Graphs in case of conductor c=1

Let $K$ be an unramified extension of $\mathbb{Q}_p$ of degree $f$, $q := p^f$ , $E$ a finite extension of $\mathbb{Q}_p$ of the ramification degree $e$ such that there is an embedding $\iota : K \hookrightarrow E$. Also, let us denote by $\mathcal{O}_E$ the ring of integers of $E$, with $\mathcal{O}_K$ the ring of integers of $K$ and with $k_E$ and $k_K \overset{\sim}{=} \mathbb{F}_q$ their residue fields. Define $G := \mathrm{GL}_2(\mathcal{O}_K)$ and $B_1 \subset G$ the subgroup of upper triangular matrices modulo $\pi_K$, where $\pi_K$ is an uniformiser of $K$. Let us denote with $K_1 \subset G$ is the subgroup of matrices which are congruent to the identity modulo $\pi_K$. Let us define the Teichmuller character $[.] : k_E \to \mathcal{O}_E$ such that for any $a \in k_E$ we have that $[a]$ is the unique element of $\mathcal{O}_E$ such that $[a]^f = [a]$ and $[a] = a \bmod \pi_E$ (the existence and uniqueness is guaranteed by Hensel's lemma). Let $\chi : B_1 \to \mathcal{O}_E^\times$ be a character of conductor 1 such that

$$\chi(\begin{pmatrix} a & b \\ pc & d \end{pmatrix}) = \chi_1(a)\chi_2(b) = \iota([\bar{a}])^{c_{\chi_1}}\iota([\bar{b}])^{c_{\chi_2}}$$

such that $\bar{\chi}_1 \neq \bar{\chi}_2$, where $^-$is modulo $p$ map, that is $\chi := \chi_1 \otimes \chi_2$. Let us define an integer $r$ between 0 and $p^{f-1}$ such that for a $x \in \mathcal{O}_E$ we have $\chi_1\chi_2^{-1}(x) = \iota([\bar{x}])^r$ and let $r_0, ..., r_{f-1}$ be integers between 0 and $p-1$ such that $r = \sum_{i=0}^{f-1} p^i r_i$. Also we define $\chi^s := \chi_2 \otimes \chi_1$.

Let us denote by $\sigma^0(\chi^s)$ the $\mathcal{O}_E[G]$ stable lattice in $ind_{B_1}^G \chi^s$ which consists of all functions $f : G \to E$ which take values in $\mathcal{O}_E$. In what follows, we will describe a filtration of $\bar{\sigma}^0(\chi^s) := \sigma^0(\chi^s) \otimes_{\mathcal{O}_E} k_E$. In order to do that we need to introduce some $f-$tuples. One defines a set of $f-$tuples $\mathcal{P}(x_0, ..., x_{f-1})$ such that $\lambda = (\lambda_1(x_1), ..., \lambda_{f-1}(x_{f-1})) \in \mathcal{P}(x_0, ..., x_{f-1})$ is such that $\lambda_i(x_i) \in \mathbb{Z} \pm x_i$. For $f = 1$, we define $\mathcal{P}(x_0) := \{x_0, p-1-x_0\}$. For $f > 1$, $\mathcal{P}(x_0, ...x_{f-1})$ is the set of all $\lambda$ such that:

1. $\lambda_i(x_i) \in \{x_i, x_i - 1, p-2-x_i, p-1-x_i\}$;
2. if $\lambda_i(x_i) \in \{x_i, x_i - 1\}$, then $\lambda_{i+1}(x_{i+1}) \in \{x_{i+1}, p-2-x_{i+1}\}$;
3. if $\lambda_i(x_i) \in \{p-2-x_i, p-1-x_i\}$, then $\lambda_{i+1}(x_{i+1}) \in \{p-1-x_{i+1}, x_{i+1}-1\}$;

where we look at indexes modulo $f$. For $\lambda = (\lambda_1(x_1), ..., \lambda_{f-1}(x_{f-1})) \in \mathcal{P}(x_0, ..., x_{f-1})$ we define the corresponding set $J_\lambda \in \{0, ..., f-1\}$ such that $i \in J_\lambda$ if and only if $\lambda_i(x_i) \in \{p-2-x_i, p-1-x_i\}$. For $\lambda = (\lambda_1(x_1), ..., \lambda_{f-1}(x_{f-1})) \in \mathcal{P}(x_0, ..., x_{f-1})$, we define $e(\lambda)$ :

$$e(\lambda) = \frac{1}{2}(\sum_{i=0}^{f-1} p^i(x_i - \lambda_i(x_i)))$$

if $f - 1 \notin J_\lambda$ and

$$e(\lambda) = \frac{1}{2}(p^f - 1 + \sum_{i=0}^{f-1} p^i(x_i - \lambda_i(x_i)))$$

otherwise.

With $(s_0, ..., s_{f-1})$ we denote the irreducible representation $(Sym^{s_0}E^2) \otimes_E (Sym^{s_1}E^2)^{Frob}$ ... $\otimes_E (Sym^{s_{f-1}}E^2)^{Frob^{f-1}}$ of $\mathrm{GL}_2(\mathcal{O}_K)$ over $k_E$. Here $(Sym^s E^2)^{Frob^i}$ denotes the representation of $\mathrm{GL}_2(\mathcal{O}_K)$ on the vector space $\oplus_{k=0}^{s} k_E x^{s-k} y^k$ where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_K)$ acts on the following way:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x^{s-k} y^k = (\bar{a}^{p^i} x + \bar{c}^{p^i} y)^{s-k} (\bar{b}^{p^i} x + \bar{d}^{p^i} y)^k.$$

**Theorem 3.1.1.** *The irreducible subquotients of $\bar{\sigma}^0(\chi)$ and $\bar{\sigma}^0(\chi^s)$ are:*

$$(\lambda_0(r_0), ..., \lambda_{f-1}(r_{f-1})) \otimes det^{e(\lambda)(r_0,...,r_{f-1})}$$

*for $\lambda \in \mathcal{P}(x_0, ..., x_{f-1})$ such that $\lambda_i(r_i) \geq 0$ for every $i$. Using the bijection between $\mathcal{P}(x_0, ..., x_{f-1})$ and the set of all subsets of $\{0, ..., f-1\}$ ($\lambda \mapsto J_\lambda$) we denote by $\mathcal{P}_\chi$ the set of all subsets of $\{0, ..., f-1\}$ which parametrize the Jordan-Holder factors of $\bar{\sigma}^0(\chi)$ and $\bar{\sigma}^0(\chi^s)$.*

*Proof.* It follows from theorem 2.2 in [7]. □

For $J \in \mathcal{P}_\chi$ we define the set of $f-$tuples $F_J \subset \{0, ..., p-1\}^f$ for which we have that $s = (s_0, ..., s_{f-1}) \in F_J$ if each $s_j$ satisfies the following conditions

- $0 \leq s_j \leq r_j$ if $j \notin J$ and $j-1 \notin J$
- $0 \leq s_j \leq r_j - 1$ if $j \notin J$ and $j-1 \in J$
- $r_j \leq s_j \leq p-1$ if $j \in J$ and $j-1 \in J$
- $r_j + 1 \leq s_j \leq p-1$ if $j \in J$ and $j-1 \notin J$

Notice that $\{0,...,p-1\}^f$ is almost a disjoint union of $F_J$ for $J \in \mathcal{P}_\chi$, with the exception that $(r_i)_{0\leq i\leq f-1} \in F_\emptyset \cap F_{\{0,...,f-1\}}$. With $\phi$ we denote the function in $\sigma^0(\chi^s)$ which is supported on $B_1$ and such that $\phi(i_1) = 1$ for any $i_1 \in I_1$, where $I_1 \subset B_1$ is the subgroup of unipotent matrices modulo $\pi_K$ of $G$. For $0 \leq j \leq q-1$ we denote $f_j := \sum\limits_{\lambda\in k_K} \iota([\lambda])^j \begin{pmatrix} [\lambda] & 1 \\ 1 & 0 \end{pmatrix} \phi$. We define:

$$\sigma_\emptyset := \Big( \bigoplus_{(s_j)_j \in F_\emptyset \setminus \{(r_j)_j\}} \mathcal{O}_E f_{\sum\limits_j p^j s_j} \Big) \oplus \mathcal{O}_E \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} f_0,$$

$$\sigma_{\{0,...,f-1\}} := \Big( \bigoplus_{(s_j)_j \in F_{\{0,...,f-1\}} \setminus \{(r_j)_j, (p-1,...,p-1)\}} \mathcal{O}_E f_{\sum\limits_j p^j s_j} \Big) \oplus \mathcal{O}_E \phi \oplus \mathcal{O}_E \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \phi,$$

$$\sigma_J := \bigoplus_{(s_j)_j \in F_J} \mathcal{O}_E f_{\sum\limits_j p^j s_j} \text{ for } J \neq \emptyset, \{0,...,f-1\}.$$

We actually have that $\sigma^0(\chi^s) = \bigoplus\limits_{J\in\mathcal{P}_\chi} \sigma_J$ and that $\phi, f_0, ..., f_{q-1}$ defines a system of compatible bases defined in 2.1.3. Also, for any $J \in \mathcal{P}_\chi$ we have that $\sigma_J \otimes_{\mathcal{O}_E} k_E$ is the corresponding Jordan-Holder factor of $\bar{\sigma}^0(\chi^s)$ as described in 3.1.1.

We study all $\mathcal{O}_E[G]$ stable lattices in $ind_{B_1}^G \chi^s$ and its corresponding graph. Let us denote by $\mathcal{G}_e^f$ the corresponding graph, as defined in 2.2.2. Since $\chi$ is trivial on $K_1$ and $K_1$ is normal in $G$ (so $K_1$ acts trivially in $ind_{B_1}^G \chi^s$),we can see $ind_{B_1}^G \chi^s$ as a representation of the finite group $G/K_1$. This together with theorems 2.2.5 and 2.2.6 gives us that $\mathcal{G}_e^f$ is finite and connected. For $u, v \in \mathcal{G}_e^f$ let us denote by $d(u,v)$ the directed distance between $u, v$ in $\mathcal{G}_e^f$. We define $\mathbb{Q}_E := \mathbb{Z}[\frac{1}{val_E(p)}]$.

Breuil in his work [6] defines a family of tuples $(v_J)_{J\subseteq\{0,...,f-1\}}$ in the following way:

**L 1.**

- *$v_J \in \mathbb{Q}_E$ and $v_\emptyset = 0$;*
- *if $J \subseteq J'$ we have that $0 \leq v_{J'} - v_J \leq |J' \setminus J|$.*

In [6] corollary 2.7 the author shows that all $\mathcal{O}_E[G]$ stable lattices in $ind_{B_1}^G \chi^s$ up to homothety look like

$$\bigoplus_{J \subseteq \{0,\dots,f-1\}} \sigma_J p^{v_J} \tag{3.1.1}$$

where $(v_J)_{J \subseteq \{0,\dots,f-1\}}$ belongs to the family of tuples defined above and $\sigma_J$ is as defined above. Notice that this implies that all $\mathcal{O}_E[G]$ stable lattices in $ind_{B_1}^G \chi^s$ look like in 3.1.1 where

**L 2.**

- *$v_J \in \mathbb{Q}_E$;*
- *if $J \subseteq J'$ we have that $0 \leq v_{J'} - v_J \leq |J' \setminus J|$.*

According to Breuil's work [6], the graph $\mathcal{G}_e^f$ does not depend on $p$, so the notation for the graph is fine.

**Theorem 3.1.2.** *Let $\Lambda_1 = \bigoplus_{J \subseteq \{0,\dots,f-1\}} \sigma_J p^{v_J^1}$ and $\Lambda_2 = \bigoplus_{J \subseteq \{0,\dots,f-1\}} \sigma_J p^{v_J^2}$ be two stable lattices in $ind_{B_1}^G \chi^s$ which belong to different homothety classes. We may assume that they are scaled so that $\Lambda_2 \subseteq \Lambda_1$ and $\Lambda_2 \not\subseteq \pi_E \Lambda_1$, since we are working with lattices up to homothety. We have that there is an edge from (the class of) $\Lambda_1$ to $\Lambda_2$ if and only if there exists $J_1 \subseteq \{0, \dots, f-1\}$ such that $v_{J_1}^1 + \frac{1}{val_E(p)} = v_{J_1}^2$ and $v_J^1 = v_J^2$ for $J \neq J_1$.*

*Proof.* As mentioned above in 2, we have that $v_J^i \in \mathbb{Q}_E$ and for any $J \subseteq J'$ we have that $0 \leq v_{J'}^i - v_J^i \leq |J' \setminus J|$, where $i = 1, 2$.

Let us first assume that there exists $J_1 \subseteq \{0, ..., f-1\}$ such that $v^1_{J_1} + \frac{1}{val_E(p)} = v^2_{J_1}$ and $v^1_J = v^2_J$, for $J \neq J_1$. This implies that $\Lambda_1/\Lambda_2 \cong \pi_E^{ev^1_{J_1}} \sigma_{J_1}/\pi_E^{ev^2_{J_1}} \sigma_{J_1} \cong \sigma_{J_1}/\pi_E^{e\frac{1}{val_E(p)}} \sigma_{J_1} \cong \sigma_{J_1}/\pi_E \sigma_{J_1} \cong \bar{\sigma}_{J_1}$, which shows that there is an is an edge from $\Lambda_1$ to $\Lambda_2$.

Let us now show the other direction. Assume that there is an edge from (the class of) $\Lambda_1$ to $\Lambda_2$. The condition $\Lambda_2 \subseteq \Lambda_1$ and $\Lambda_2 \not\subset \pi_E \Lambda_1$ together with the fact $[\Lambda_1] \neq [\Lambda_2]$ implies that

- $v^1_J \leq v^2_J$ for all $J \subset \{0, ..., f-1\}$;
- there exists $J_1 \subseteq \{0, ..., f-1\}$ such that $v^1_{J_1} < v^2_{J_1}$;
- there is $J_{eq} \subseteq \{0, ..., f-1\}$ such that $v^1_{J_{eq}} = v^2_{J_{eq}}$.

We will show that $J_1$ is unique and that $v^2_{J_1} = v^1_{J_1} + \frac{1}{val_E(p)}$. In particular, we show it via contradiction. First, let us assume that there is $J_2 \subseteq \{0, ..., f-1\}$ such that $J_2 \neq J_1$ and $v^1_{J_2} < v^2_{J_2}$. This implies that $\bar{\sigma}_{J_1}$ and $\bar{\sigma}_{J_2}$ appear as Jordan Holder factors in $\Lambda_1/\Lambda_2$, which implies that $\Lambda_1/\Lambda_2$ is not irreducible. But this contradicts the fact that there is an edge from $\Lambda_1$ to $\Lambda_2$. In order to show that $v^2_{J_1} = v^1_{J_1} + \frac{1}{val_E(p)}$ let us assume otherwise, that is let us assume that $v^2_{J_1} = v^1_{J_1} + \frac{k}{val_E(p)}$ for an integer $k > 1$. This implies that $\Lambda_1/\Lambda_2 \cong \sigma_{J_1}/\pi_E^{e\frac{k}{val_E(p)}} \sigma_{J_1} \cong \sigma_{J_1}/\pi_E^k \sigma_{J_1}$. But we have $\sigma_{J_1}/\pi_E^k \sigma_{J_1} \twoheadrightarrow \sigma_{J_1}/\pi_E \sigma_{J_1}$ is surjective and not injective, which implies that $\sigma_{J_1}/\pi_E^k \sigma_{J_1}$ is not irreducible. This contradicts the fact that there is an edge from $\Lambda_1$ to $\Lambda_2$. This finishes our proof. $\square$

**Theorem 3.1.3.** *There is an injection $\mu : \mathcal{G}^f_e \hookrightarrow \mathcal{G}^f_{e+1}$.*

*Proof.* Again, we will be using the description of lattices via tuples 2. Let $\Lambda_e = \sum\limits_{J \subset \{0,...,f-1\}} \pi_E^{e\frac{i_J}{e}} \sigma_J \in \mathcal{G}^f_e$ and let us define $\mu(\Lambda_e) = \sum\limits_{J \subset \{0,...,f-1\}} \pi_E^{e\frac{i_J}{e+1}} \sigma_J$. Let us first show that $\Lambda_{e+1} := \mu(\Lambda_e) \in \mathcal{G}^f_{e+1}$. Using 2, we only need to show that for $J \subset J' \subset \{0, ..., f-1\}$ we have that $0 \leq \frac{i_{J'}}{e+1} - \frac{i_J}{e+1} \leq |J' \setminus J|$. But since $\Lambda_e \in \mathcal{G}^f_e$ we have that $0 \leq \frac{i_{J'}}{e} - \frac{i_J}{e} \leq |J' \setminus J|$, which implies the previous inequality by multiplying with $\frac{e}{e+1} < 1$. To finish the proof we need to show that for $\Lambda_e, \Lambda_e^{'}$ such that $d(\Lambda_e, \Lambda_e^{'}) = 1$, we have that $d(\mu(\Lambda_e), \mu(\Lambda_e^{'})) = 1$. Let $\Lambda_e = \sum\limits_{J \subset \{0,...,f-1\}} \pi_E^{e^{\frac{i_J}{e}}} \sigma_J \in \mathcal{G}^f_e$ and $\Lambda_e^{'} = \sum\limits_{J \subset \{0,...,f-1\}} \pi_E^{e^{\frac{j_J}{e}}} \sigma_J \in \mathcal{G}^f_e$. Since $d(\Lambda_e, \Lambda_e^{'}) = 1$ we have that there is $J_{\Lambda_e, \Lambda_e^{'}} \subset \{0, ..., f-1\}$ such that $i_J = j_J$ for any $J \neq J_{\Lambda_e, \Lambda_e^{'}}$, $J \subset \{0, ..., f-1\}$ and $j_{J_{\Lambda_e, \Lambda_e^{'}}} = i_{J_{\Lambda_e, \Lambda_e^{'}}} + 1$. But this implies that $d(\mu(\Lambda_e), \mu(\Lambda_e^{'})) = 1$. $\square$

**Theorem 3.1.4.** *The number of nodes of the graph $\mathcal{G}^2_e$ is $\frac{e+1}{3}(2e^2 + 4e + 3)$.*

*Proof.* In the case when $f = 2$ we have that $v_\emptyset$, $v_{\{0\}}$, $v_{\{1\}}$ and $v_{\{0,1\}}$ which satisfy 1 and determine a lattice. Since $v_\emptyset = 0$, we have that $v_{\{0\}}$ and $v_{\{1\}}$ can take any value between 0 and 1, that is $v_{\{0\}}$ and $v_{\{1\}}$ can take any value from $\{0, \frac{1}{e}, ... \frac{e-1}{e}, 1\}$. For a given $v_{\{0\}}$ and $v_{\{1\}}$ the coefficient $v_{\{0,1\}}$ can take any value which satisfies $max(v_{\{0\}}, v_{\{1\}}) \leq v_{\{0,1\}} \leq min(v_{\{0\}}, v_{\{1\}}) + 1$, which is possible on $(1 + min(v_{\{0\}}, v_{\{1\}}) - max(v_{\{0\}}, v_{\{1\}}))e + 1$ ways. Let us denote $v_1 := v_{\{0\}}$ and $v_2 := v_{\{1\}}$. The preceding discussion implies that

$$
\begin{aligned}
|\mathcal{G}^2_e| &= \sum_{v_1, v_2 \in \{0, \frac{1}{e}, ... \frac{e-1}{e}, 1\}} ((1 + min(v_1, v_2) - max(v_1, v_2))e + 1) \\
&= \sum_{0 \leq n_1, n_2 \leq e} (|e + min(n_1, n_2) - max(n_1, n_2)| + 1)
\end{aligned}
$$

where $n_1 = ev_1$ and $n_2 := ev_2$. We denote $n_{min} := min(n_1, n_2)$ and $d := max(n_1, n_2) - min(n_1, n_2)$. We substitute this in the sum and we split in the cases when $n_1 \neq n_2$

and $n_1 = n_2$:

$$\begin{aligned}
|\mathcal{G}_e^2| &= 2 \sum_{0 \le n_{min} \le e} ( \sum_{0 < d \le e - n_{min}} (|e-d|+1)) + \sum_{0 \le n \le e} (|e|+1) \\
&= ( \sum_{0 \le n_{min} \le e} 2(e+1)(e-n_{min}) - (e-n_{min})(e-n_{min}+1)) + (e+1)^2 \\
&= ( \sum_{0 \le n_{min} \le e} e^2 + e - n_{min} - n_{min}^2) + (e+1)^2 \\
&= (e^2(e+1) + e(e+1) - \frac{e(e+1)}{2} - \frac{e(e+1)(2e+1)}{6}) + (e+1)^2 \\
&= \frac{1}{3}(e+1)(2e^2+4e+3).
\end{aligned}$$

□

Until the end of the section, we will be looking into the associated graphs, which we obtain using the results above. In the case when $f = 1$ and $e = 1$ the set of lattices is parametrised with $v_\emptyset = 0$, $v_{\{0\}} = 0$ and $v_\emptyset = 0$, $v_{\{0\}} = 1$. When $f = 1$, $e = 1$ the graph is shown in the figure 3.1.


**Figure 3.1:** $f = 1$, $e = 1$

In the case when $f = 2$, $e = 1$ the graph is shown in the figure 3.2.

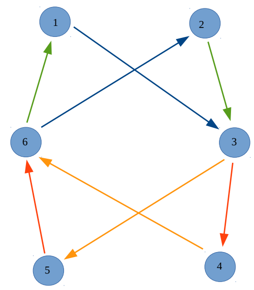


**Figure 3.2:** $f = 2$, $e = 1$

In the case when $f = 3$, $e = 1$ the graph is shown in the figure 3.3.

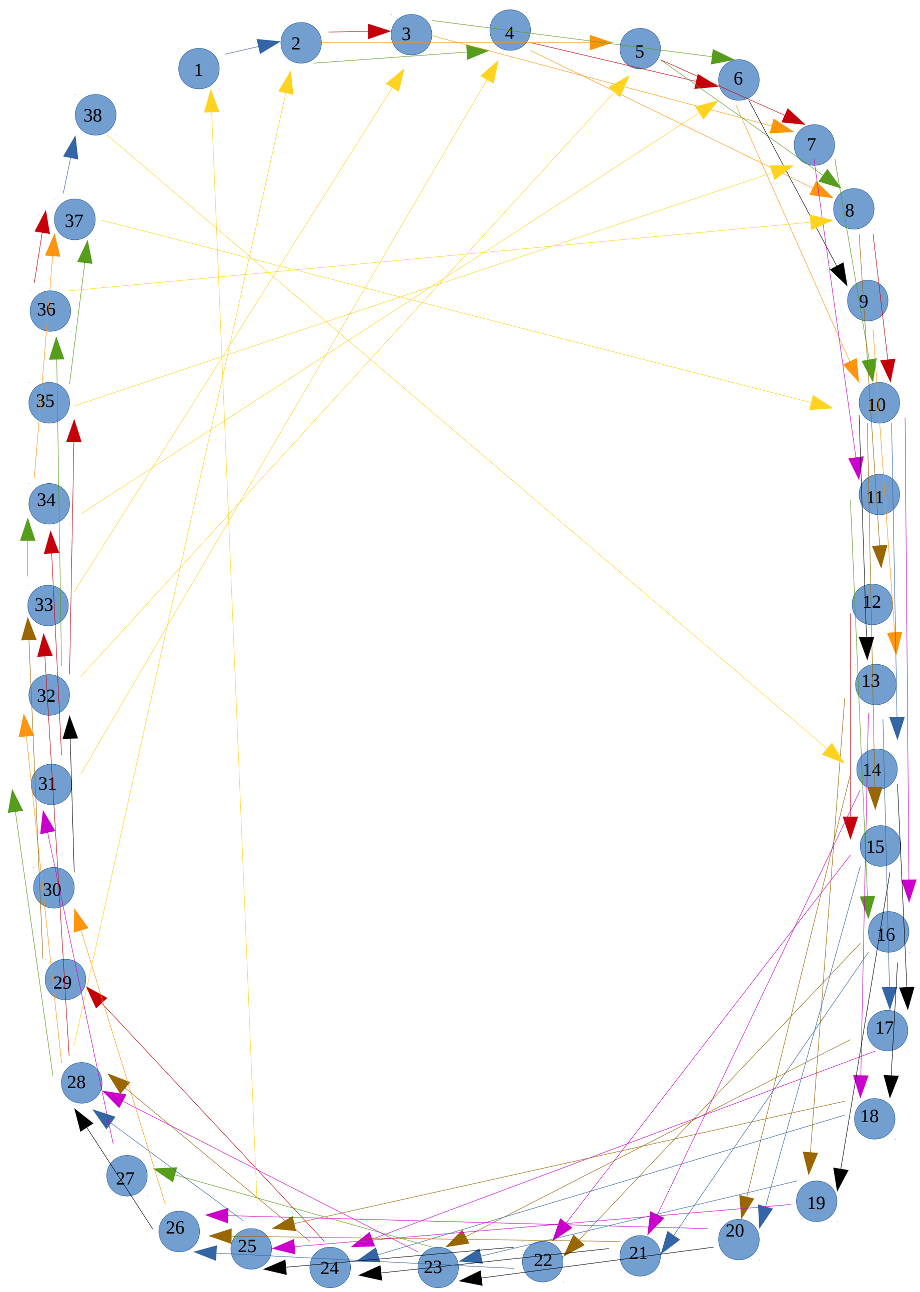


**Figure 3.3:** $f = 3$, $e = 1$

In the case when $f = 2$, $e = 2$ the graph is shown in the figure 3.4.

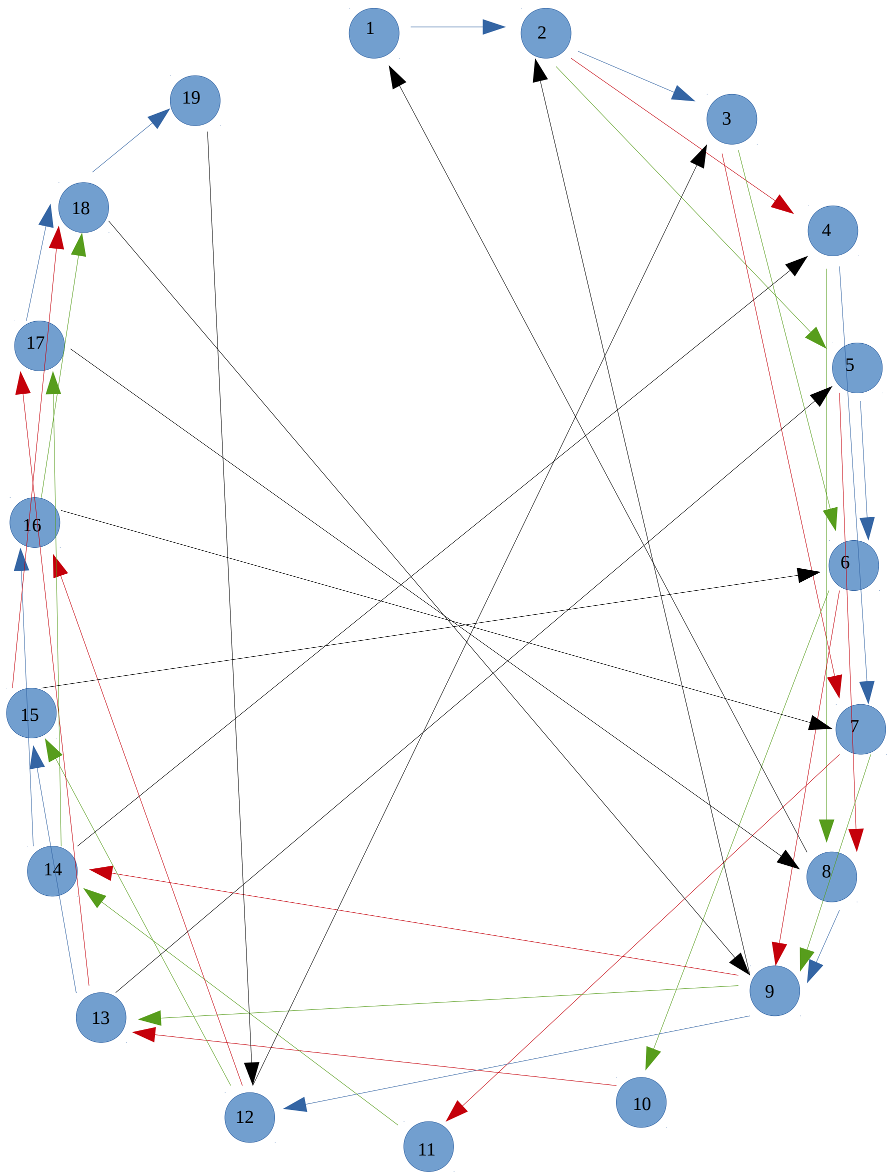


**Figure 3.4:** $f = 2$, $e = 2$

## 3.2 Proof of the bound in non-tame principal series type representations of $\mathrm{GL}_2(\mathcal{O}_F)$ over the $p$-adics

In this section we are studying lattices in the family of non-tame principal series type representations $ind_{B_n}^G \chi$, where $G := \mathrm{GL}_2(\mathcal{O}_F)$, where $F/\mathbb{Q}_p$ is finite, $B_n \subset G$ subgroup of upper-triangular matrices modulo $\pi_F^n$, $n \geq 2$ and $\chi := \chi_1 \otimes \chi_2$ a character of $B_n$ over some extension $E$ of $\mathbb{Q}_p$, where each $\chi_1$, $\chi_2$ are characters of $\mathcal{O}_F^\times$ trivial on $1 + \pi_F^n \mathcal{O}_F$ and non trivial on $1 + \pi_F^{n-1}\mathcal{O}_F$. We also assume that $E$ contains the maximal unramified subfield of $F$. Let us also $k_F := \mathcal{O}_F/\pi_F\mathcal{O}_F$, $|k_F| = q = p^f$. We have that the dimension of the representation is $q^{n-1}(q+1)$. Let us define $\zeta := \chi_1^{-1}\chi_2$. We have that $\zeta$ be a multiplicative character of $\mathcal{O}_F^\times$, trivial on $1 + \pi_F^n\mathcal{O}_F$ and we will assume non trivial on $1 + \pi_F^{n-1}\mathcal{O}_F$, since we are interesed in the case when $ind_{B_n}^G\chi$ is irreducible. For $\lambda \in \mathbb{F}_q^\times$ we denote by $[\lambda]$ the Teichmuller lift of $\lambda$. Let us also denote $0 \leq c_1, c_2 < q - 1$ such that $\chi_i([\lambda]) = [\lambda]^{c_i}$. We denote $c := c_1 - c_2$. We write $\phi_{I_n} \in ind_{B_n}^G \chi$ for the function supported on $B_n$ such that $\phi_{I_n}(i_n) = 1$ for $i_n \in I_n$, where $I_n \subset B_n$ is the subgroup of $G$ of matrices congruent to $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ modulo $\pi_F^n$.

Let

$$N_\gamma := \sum_{\omega \in \pi_F\mathcal{O}_F/\pi_F^n\mathcal{O}_F} \zeta(1+\gamma\omega) \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix} \phi_{I_n},\ \gamma \in \mathcal{O}_F/\pi_F^{n-1} \tag{3.2.1}$$

$$N^\gamma := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} N_\gamma = \sum_{\omega \in \pi_F\mathcal{O}_F/\pi_F^n\mathcal{O}_F} \zeta(1+\gamma\omega) \begin{pmatrix} \omega & 1 \\ 1 & 0 \end{pmatrix} \phi_{I_n},\ \gamma \in \mathcal{O}_F/\pi_F^{n-1}. \tag{3.2.2}$$

Then we have for $t, m \in \mathcal{O}_F$ that

$$
\begin{aligned}
\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} N_\gamma &= \sum_{\omega \in \pi_F \mathcal{O}_F / \pi_F^n \mathcal{O}_F} \zeta(1+\gamma\omega) \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix} \phi_{I_n} \\
&= \sum_{\omega \in \pi_F \mathcal{O}_F / \pi_F^n \mathcal{O}_F} \zeta(1+\gamma\omega) \begin{pmatrix} 1+\omega t & t \\ \omega & 1 \end{pmatrix} \phi_{I_n} \\
&= \sum_{\omega \in \pi_F \mathcal{O}_F / \pi_F^n \mathcal{O}_F} \zeta(1+\gamma\omega) \begin{pmatrix} 1 & 0 \\ \frac{\omega}{1+t\omega} & 1 \end{pmatrix} \begin{pmatrix} 1+t\omega & t \\ 0 & 1-\frac{t\omega}{1+\omega t} \end{pmatrix} \phi_{I_n} \\
&= \sum_{\omega \in \pi_F \mathcal{O}_F / \pi_F^n \mathcal{O}_F} \zeta(1+\gamma\omega) \chi_1\chi_2^{-1}(1+\omega t) \begin{pmatrix} 1 & 0 \\ \frac{\omega}{1+t\omega} & 1 \end{pmatrix} \phi_{I_n} \\
&= \sum_{\kappa \in \pi_F \mathcal{O}_F / \pi_F^n \mathcal{O}_F} \zeta(1+\frac{\gamma\kappa}{1-t\kappa}) \zeta^{-1}(\frac{1}{1-t\kappa}) \begin{pmatrix} 1 & 0 \\ \kappa & 1 \end{pmatrix} \phi_{I_n} \\
&= \sum_{\kappa \in \pi_F \mathcal{O}_F / \pi_F^n \mathcal{O}_F} \zeta(1+\kappa(\gamma-t)) \zeta(\frac{1}{1-t\kappa}) \zeta^{-1}(\frac{1}{1-t\kappa}) \begin{pmatrix} 1 & 0 \\ \kappa & 1 \end{pmatrix} \phi_{I_n} \\
&= \sum_{\kappa \in \pi_F \mathcal{O}_F / \pi_F^n \mathcal{O}_F} \zeta(1+\kappa(\gamma-t)) \begin{pmatrix} 1 & 0 \\ \kappa & 1 \end{pmatrix} \phi_{I_n} \\
&= N_{\gamma-t}.
\end{aligned}
\tag{3.2.3}
$$

$$\begin{aligned}
\begin{pmatrix} 1 & 0 \\ \pi_F m & 1 \end{pmatrix} N_\gamma &= \begin{pmatrix} 1 & 0 \\ \pi_F m & 1 \end{pmatrix} \sum_{\omega \in \pi_F \mathcal{O}_F / \pi_F^n \mathcal{O}_F} \zeta(1+\gamma\omega) \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix} \phi_{I_n} \\
&= \sum_{\omega \in \pi_F \mathcal{O}_F / \pi_F^n \mathcal{O}_F} \zeta(1+\gamma\omega) \begin{pmatrix} 1 & 0 \\ \pi_F m + \omega & 1 \end{pmatrix} \phi_{I_n} \\
&= \sum_{\kappa \in \pi_F \mathcal{O}_F / \pi_F^n \mathcal{O}_F} \zeta(1+\gamma(\kappa - \pi_F m)) \begin{pmatrix} 1 & 0 \\ \kappa & 1 \end{pmatrix} \phi_{I_n} \\
&= \sum_{\kappa \in \pi_F \mathcal{O}_F / \pi_F^n \mathcal{O}_F} \zeta(1-\gamma\pi_F m)\zeta(1+\frac{\gamma\kappa}{1-\gamma\pi_F m}) \begin{pmatrix} 1 & 0 \\ \kappa & 1 \end{pmatrix} \phi_{I_n} \\
&= \zeta(1-\gamma\pi_F m) N_{\frac{\gamma}{1-\gamma\pi_F m}}.
\end{aligned} \tag{3.2.4}$$

Let $a, d \in \mathcal{O}_F^\times$. We also have that

$$\begin{aligned}
\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} N_\gamma &= \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \sum_{\omega \in \pi_F \mathcal{O}_F / \pi_F^n \mathcal{O}_F} \zeta(1+\gamma\omega) \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix} \phi_{I_n} \\
&= \sum_{\omega \in \pi_F \mathcal{O}_F / \pi_F^n \mathcal{O}_F} \zeta(1+\gamma\omega) \begin{pmatrix} a & 0 \\ d\omega & d \end{pmatrix} \phi_{I_n} \\
&= \sum_{\omega \in \pi_F \mathcal{O}_F / \pi_F^n \mathcal{O}_F} \zeta(1+\gamma\omega) \begin{pmatrix} 1 & 0 \\ \frac{d}{a}\omega & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \phi_{I_n} \\
&= \chi_1(a)\chi_2(d) \sum_{\omega \in \pi_F \mathcal{O}_F / \pi_F^n \mathcal{O}_F} \zeta(1+\gamma\omega\frac{a}{d}) \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix} \phi_{I_n} \\
&= \chi_1(a)\chi_2(d) N_{\gamma\frac{a}{d}}.
\end{aligned} \tag{3.2.5}$$

Let us also define

$$N_{i,\gamma} := \sum_{\lambda \in \mathbb{F}_q^\times} [\lambda]^i \begin{pmatrix} [\lambda] & 1 \\ 1 & 0 \end{pmatrix} N_\gamma$$

where $0 \leq i < q-1$, $\gamma \in \mathcal{O}_F/\pi_F^{n-1}\mathcal{O}_F$.

**Lemma 3.2.1.** *Let $\Pi_n$ be an $\mathcal{O}_E$ lattice with basis*

$$N_{i,\gamma}, \text{ where } \gamma \in \mathcal{O}_F/\pi_F^{n-1}\mathcal{O}_F, 0 \leq i < q-1$$

$$N_\gamma, \text{ where } \gamma \in \mathcal{O}_F/\pi_F^{n-1}$$

$$N^\gamma, \text{ where } \gamma \in \mathcal{O}_F/\pi_F^{n-1}\mathcal{O}_F.$$

*Then $\Pi_n$ is $\mathcal{O}_E[G]$ stable lattice. In particular, $\Pi_n$ is the smallest $\mathcal{O}_E[G]$ stable lattice containing $N_0$.*

*Proof.* Let $B_1 \subset G$ be the subgroup of upper triangular matrices modulo $\pi_F$ and let $I_1 \subset G$ be the subgroup of unipotent matrices. We have that

$$G = \bigsqcup_{\lambda \in \mathcal{O}_F/\pi_F\mathcal{O}_F} \begin{pmatrix} [\lambda] & 1 \\ 1 & 0 \end{pmatrix} B_1 \bigsqcup B_1$$

$$= \bigcup_{\substack{a,d,\lambda \in \mathbb{F}_q^\times \\ m,t \in \mathcal{O}_F}} \begin{pmatrix} [\lambda] & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \pi_F m & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$\bigcup_{\substack{a,d \in \mathbb{F}_q^\times \\ m,t \in \mathcal{O}_F}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \pi_F m & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$\bigcup_{\substack{a,d \in \mathbb{F}_q^\times \\ m,t \in \mathcal{O}_F}} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \pi_F m & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Having in mind 3.2, 3.2.4 and 3.2.3 we have that the set

$$\{N_\gamma, N^\gamma, \begin{pmatrix} [\lambda] & 1 \\ 1 & 0 \end{pmatrix} N_\gamma \mid \gamma \in \mathcal{O}_F/\pi_F^{n-1}\mathcal{O}_F, \lambda \in \mathbb{F}_q^\times\}$$

generates a (finitely generated) $G$-stable $\mathcal{O}_E$- submodule $\mathcal{O}_E[G]N_0$, which is then a lattice by the irreducibility of $ind_{B_n}^G\chi$. Since there are $q^{n-1}(q+1)$ number of generators, which is also the dimension of $ind_{B_n}^G\chi$, we have that they are actually a basis of $\mathcal{O}_E[G]N_0$. So we have that the set

$$\{N_\gamma, N^\gamma, \begin{pmatrix} [\lambda] & 1 \\ 1 & 0 \end{pmatrix} N_\gamma \mid \gamma \in \mathcal{O}_F/\pi_F^{n-1}\mathcal{O}_F, \lambda \in \mathbb{F}_q^\times\}$$

is the basis of $\mathcal{O}_E[G]N_0$, that is

$$\mathcal{O}_E[G]N_0 = \bigoplus_{\substack{\lambda\in\mathbb{F}_q^\times \\ \gamma\in\mathcal{O}_F/\pi_F^{n-1}\mathcal{O}_F}} \mathcal{O}_E \begin{pmatrix} [\lambda] & 1 \\ 1 & 0 \end{pmatrix} N_\gamma \bigoplus_{\gamma\in\mathcal{O}_F/\pi_F^{n-1}\mathcal{O}_F} \mathcal{O}_E N_\gamma + \bigoplus_{\gamma\in\mathcal{O}_F/\pi_F^{n-1}\mathcal{O}_F} \mathcal{O}_E N^\gamma.$$

Recall that $N_{i,\gamma} = \sum_{\lambda\in\mathbb{F}_q^\times} [\lambda]^i \begin{pmatrix} [\lambda] & 1 \\ 1 & 0 \end{pmatrix} N_\gamma$. Since the matrix $\left([\lambda]^i\right)_{\substack{0\le i<q-1 \\ \lambda\in\mathbb{F}_q^\times}}$ is invertible, we actually have that for any $\gamma \in \mathcal{O}_F/\pi_F^{n-1}\mathcal{O}_F$,

$$\bigoplus_{\lambda\in\mathbb{F}_q^\times} \mathcal{O}_E \begin{pmatrix} [\lambda] & 1 \\ 1 & 0 \end{pmatrix} N_\gamma = \bigoplus_{0\le i<q-1} \mathcal{O}_E N_{i,\gamma}.$$

This finally gives that $\mathcal{O}_E[G]N_0 = \mathcal{O}_E[G]N_j = \Pi_n$. □

Let us define two operators and let us see how they act on the basis $(N_{i,\gamma}, N_\gamma, N^\gamma)$ of lattice $\Pi_n$.

$$\mathbb{O}_s^T := \sum_{\omega\in\pi_F^{n-1}\mathcal{O}_F/\pi_F^n\mathcal{O}_F} \zeta(1+s\omega) \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix},$$

for $s \in \mathcal{O}_F/\pi_F\mathcal{O}_F$ and

$$\mathbb{O}_s^D := \sum_{\omega \in \pi_F^{n-1}\mathcal{O}_F/\pi_F^n\mathcal{O}_F} \zeta(1+s\omega)\chi_1^{-1}(1+\omega) \begin{pmatrix} 1+\omega & 0 \\ 0 & 1 \end{pmatrix},$$

for $s \in \mathcal{O}_F/\pi_F\mathcal{O}_F$.

Now we will compute the action of operators $\mathbb{O}_s^T$ and $\mathbb{O}_s^D$ on $N_{i,\gamma}$, $N_\gamma$ and $N^\gamma$ since we need it for the computation afterwards. We have that $\mathbb{O}_s^T$ acts on $N_{i,\gamma}$ as:

$$\begin{aligned}
\mathbb{O}_s^T N_{i,\gamma} &= \sum_{\substack{\lambda_1 \in \mathbb{F}_q^\times \\ \lambda_2 \in \pi_F\mathcal{O}_F/\pi_F^n\mathcal{O}_F \\ \omega \in \pi_F^{n-1}\mathcal{O}_F/\pi_F^n\mathcal{O}_F}} \zeta(1+s\omega)[\lambda_1]^i \zeta(1+\gamma\lambda_2) \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix} \begin{pmatrix} [\lambda_1]+\lambda_2 & 1 \\ 1 & 0 \end{pmatrix} \phi_{I_n} \\
&= \sum_{\substack{\lambda_1 \in \mathbb{F}_q^\times \\ \lambda_2 \in \pi_F\mathcal{O}_F/\pi_F^n\mathcal{O}_F \\ \omega \in \pi_F^{n-1}\mathcal{O}_F/\pi_F^n\mathcal{O}_F}} \zeta(1+s\omega)[\lambda_1]^i \zeta(1+\gamma\lambda_2) \begin{pmatrix} [\lambda_1]+\lambda_2+\omega & 1 \\ 1 & 0 \end{pmatrix} \phi_{I_n} \\
&= \sum_{\substack{\lambda_1 \in \mathbb{F}_q^\times \\ \kappa \in \pi_F\mathcal{O}_F/\pi_F^n\mathcal{O}_F}} \Bigg( \sum_{\omega \in \pi_F^{n-1}\mathcal{O}_F/\pi_F^n\mathcal{O}_F} \zeta(1+\omega(s-\gamma)) \Bigg) \zeta(1+\gamma\kappa)[\lambda_1]^i \begin{pmatrix} [\lambda_1]+\kappa & 1 \\ 1 & 0 \end{pmatrix} \phi_{I_n} \\
&= \Bigg( \sum_{\omega \in \pi_F^{n-1}\mathcal{O}_F/\pi_F^n\mathcal{O}_F} \zeta(1+\omega(s-\gamma)) \Bigg) \sum_{\substack{\lambda_1 \in \mathbb{F}_q^\times \\ \kappa \in \pi_F\mathcal{O}_F/\pi_F^n\mathcal{O}_F}} \zeta(1+\gamma\kappa)[\lambda_1]^i \begin{pmatrix} [\lambda_1]+\kappa & 1 \\ 1 & 0 \end{pmatrix} \phi_{I_n} \\
&= \Bigg( \sum_{\omega \in \pi_F^{n-1}\mathcal{O}_F/\pi_F^n\mathcal{O}_F} \zeta(1+\omega(s-\gamma)) \Bigg) N_{i,\gamma}.
\end{aligned}$$

We have that the value above is $qN_{i,\gamma}$ if $s = \gamma \bmod \pi_F$ and 0 otherwise.

We now compute the action of $\mathbb{O}_s^T$ on $N_\gamma$:

$$\begin{aligned}
\mathbb{O}_s^T N_\gamma &= \sum_{\substack{\lambda\in\pi_F\mathcal{O}_F/\pi_F^n\mathcal{O}_F\\ \omega\in\pi_F^{n-1}\mathcal{O}_F/\pi_F^n\mathcal{O}_F}} \zeta(1+s\omega)\zeta(1+\gamma\lambda)\begin{pmatrix}1 & \omega\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \lambda & 1\end{pmatrix}\phi_{I_n}\\
&= \sum_{\substack{\lambda\in\pi_F\mathcal{O}_F/\pi_F^n\mathcal{O}_F\\ \omega\in\pi_F^{n-1}\mathcal{O}_F/\pi_F^n\mathcal{O}_F}} \zeta(1+s\omega)\zeta(1+\gamma\lambda)\begin{pmatrix}1+\omega\lambda & \omega\\ \lambda & 1\end{pmatrix}\phi_{I_n}\\
&= \sum_{\substack{\lambda\in\pi_F\mathcal{O}_F/\pi_F^n\mathcal{O}_F\\ \omega\in\pi_F^{n-1}\mathcal{O}_F/\pi_F^n\mathcal{O}_F}} \zeta(1+s\omega)\zeta(1+\gamma\lambda)\begin{pmatrix}1 & 0\\ \frac{\lambda}{1+\omega\lambda} & 1\end{pmatrix}\begin{pmatrix}1+\omega\lambda & \omega\\ 0 & 1-\frac{\omega\lambda}{1+\omega\lambda}\end{pmatrix}\phi_{I_n}\\
&= \sum_{\substack{\lambda\in\pi_F\mathcal{O}_F/\pi_F^n\mathcal{O}_F\\ \omega\in\pi_F^{n-1}\mathcal{O}_F/\pi_F^n\mathcal{O}_F}} \zeta(1+s\omega)\zeta(1+\gamma\lambda)\chi_1\chi_2^{-1}(1+\omega\lambda)\begin{pmatrix}1 & 0\\ \frac{\lambda}{1+\omega\lambda} & 1\end{pmatrix}\phi_{I_n}\\
&= \sum_{\substack{\kappa\in\pi_F\mathcal{O}_F/\pi_F^n\mathcal{O}_F\\ \omega\in\pi_F^{n-1}\mathcal{O}_F/\pi_F^n\mathcal{O}_F}} \zeta(1+s\omega)\zeta(1+\gamma\kappa)\begin{pmatrix}1 & 0\\ \kappa & 1\end{pmatrix}\phi_{I_n}
\end{aligned}$$

where in the last line we used that $\kappa\omega = 0 \bmod \pi_F^n$. So we have

$$\mathbb{O}_s^T N_\gamma = \sum_{\kappa\in\pi_F\mathcal{O}_F/\pi_F^n\mathcal{O}_F}\Bigg(\sum_{\omega\in\pi_F^{n-1}\mathcal{O}_F/\pi_F^n\mathcal{O}_F}\zeta(1+s\omega)\Bigg)\zeta(1+\kappa\gamma)\begin{pmatrix}1 & 0\\ \kappa & 1\end{pmatrix}\phi_{I_n}$$

This is equal to 0 if $s \neq 0$ and $qN_\gamma$ otherwise. Let us now compute the action of $\mathbb{O}_s^T$

on $N^\gamma$:

$$\begin{aligned}
\mathbb{O}_s^T N^\gamma &= \sum_{\substack{\lambda\in\pi_F\mathcal{O}_F/\pi_F^n\mathcal{O}_F \\ \omega\in\pi_F^{n-1}\mathcal{O}_F/\pi_F^n\mathcal{O}_F}} \zeta(1+s\omega)\zeta(1+\gamma\lambda)\begin{pmatrix}1 & \omega\\ 0 & 1\end{pmatrix}\begin{pmatrix}\lambda & 1\\ 1 & 0\end{pmatrix}\phi_{I_n} \\
&= \sum_{\substack{\lambda\in\pi_F\mathcal{O}_F/\pi_F^n\mathcal{O}_F \\ \omega\in\pi_F^{n-1}\mathcal{O}_F/\pi_F^n\mathcal{O}_F}} \zeta(1+s\omega)\zeta(1+\gamma\lambda)\begin{pmatrix}\omega+\lambda & 1\\ 1 & 0\end{pmatrix}\phi_{I_n} \\
&= \sum_{\substack{\kappa\in\pi_F\mathcal{O}_F/\pi_F^n\mathcal{O}_F \\ \omega\in\pi_F^{n-1}\mathcal{O}_F/\pi_F^n\mathcal{O}_F}} \zeta(1+s\omega)\zeta(1+\gamma(\kappa-\omega))\begin{pmatrix}\kappa & 1\\ 1 & 0\end{pmatrix}\phi_{I_n} \\
&= \sum_{\substack{\kappa\in\pi_F\mathcal{O}_F/\pi_F^n\mathcal{O}_F \\ \omega\in\pi_F^{n-1}\mathcal{O}_F/\pi_F^n\mathcal{O}_F}} \zeta(1+s\omega)\zeta(1+\gamma\kappa)\zeta(1-\gamma\omega)\begin{pmatrix}\kappa & 1\\ 1 & 0\end{pmatrix}\phi_{I_n} \\
&= \sum_{\kappa\in\pi_F\mathcal{O}_F/\pi_F^n\mathcal{O}_F}\Bigg(\sum_{\omega\in\pi_F^{n-1}\mathcal{O}_F/\pi_F^n\mathcal{O}_F}\zeta(1+(s-\gamma)\omega)\Bigg)\zeta(1+\gamma\kappa)\begin{pmatrix}\kappa & 1\\ 1 & 0\end{pmatrix}\phi_{I_n} \\
&= \Bigg(\sum_{\omega\in\pi_F^{n-1}\mathcal{O}_F/\pi_F^n\mathcal{O}_F}\zeta(1+(s-\gamma)\omega)\Bigg)N^\gamma
\end{aligned}$$

This is equal to 0 if $s \neq \gamma \mod \pi_F$ and $qN^\gamma$ otherwise. This finishes the description of the action of the operator $\mathbb{O}_s^T$. We continue with looking into the action of $\mathbb{O}_s^D$. We first compute the action on $N_{i,\gamma}$:

$$\mathbb{O}_s^D N_{i,\gamma} = \sum_{\substack{\lambda_1 \in \mathbb{F}_q^\times \\ \lambda_2 \in \pi_F \mathcal{O}_F/\pi_F^n \mathcal{O}_F \\ \omega \in \pi_F^{n-1}\mathcal{O}_F/\pi_F^n\mathcal{O}_F}} \zeta(1+s\omega)\chi_1^{-1}(1+\omega)[\lambda_1]^i \zeta(1+\gamma\lambda_2) \begin{pmatrix} 1+\omega & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} [\lambda_1]+\lambda_2 & 1 \\ 1 & 0 \end{pmatrix} \phi_{I_n}$$

$$= \sum_{\substack{\lambda_1 \in \mathbb{F}_q^\times \\ \lambda_2 \in \pi_F \mathcal{O}_F/\pi_F^n \mathcal{O}_F \\ \omega \in \pi_F^{n-1}\mathcal{O}_F/\pi_F^n\mathcal{O}_F}} \zeta(1+s\omega)\chi_1^{-1}(1+\omega)[\lambda_1]^i \zeta(1+\gamma\lambda_2) \begin{pmatrix} ([\lambda_1]+\lambda_2)(1+\omega) & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1+\omega \end{pmatrix} \phi_{I_n}$$

$$= \sum_{\substack{\lambda_1 \in \mathbb{F}_q^\times \\ \lambda_2 \in \pi_F \mathcal{O}_F/\pi_F^n \mathcal{O}_F \\ \omega \in \pi_F^{n-1}\mathcal{O}_F/\pi_F^n\mathcal{O}_F}} \zeta(1+s\omega)\chi_1^{-1}(1+\omega)[\lambda_1]^i \zeta(1+\gamma\lambda_2)\chi_2(1+\omega) \begin{pmatrix} [\lambda_1]+\lambda_2+[\lambda_1]\omega+\lambda_2\omega & 1 \\ 1 & 0 \end{pmatrix} \phi_{I_n}$$

Let us define $\kappa = \lambda_2 + [\lambda_1]\omega + \lambda_2\omega$. So the line above now can be written as

$$\sum_{\substack{\lambda_1 \in \mathbb{F}_q^\times \\ \kappa \in \pi_F \mathcal{O}_F/\pi_F^n \mathcal{O}_F \\ \omega \in \pi_F^{n-1}\mathcal{O}_F/\pi_F^n\mathcal{O}_F}} \zeta(1+s\omega)\chi_1^{-1}\chi_2(1+\omega)[\lambda_1]^i \zeta(1+\gamma\frac{\kappa-[\lambda_1]\omega}{1+\omega}) \begin{pmatrix} [\lambda_1]+\kappa & 1 \\ 1 & 0 \end{pmatrix} \phi_{I_n} =$$

$$\sum_{\substack{\lambda_1 \in \mathbb{F}_q^\times \\ \kappa \in \pi_F \mathcal{O}_F/\pi_F^n \mathcal{O}_F \\ \omega \in \pi_F^{n-1}\mathcal{O}_F/\pi_F^n\mathcal{O}_F}} \zeta(1+s\omega)\chi_1^{-1}\chi_2(1+\omega))[\lambda_1]^i \zeta(1+\omega+\gamma(\kappa-[\lambda_1]\omega))\zeta^{-1}(1+\omega) \begin{pmatrix} [\lambda_1]+\kappa & 1 \\ 1 & 0 \end{pmatrix} \phi_{I_n}.$$

We continue with the computation:

$$\mathbb{O}_s^D N_{i,\gamma} = \sum_{\substack{\lambda_1 \in \mathbb{F}_q^\times \\ \kappa \in \pi_F \mathcal{O}_F/\pi_F^n \mathcal{O}_F \\ \omega \in \pi_F^{n-1}\mathcal{O}_F/\pi_F^n\mathcal{O}_F}} \zeta(1+s\omega)[\lambda_1]^i \zeta(1+(1-\gamma[\lambda_1])\omega))\zeta(1+\gamma\kappa) \begin{pmatrix} [\lambda_1]+\kappa & 1 \\ 1 & 0 \end{pmatrix} \phi_{I_n}$$

$$= \sum_{\substack{\lambda_1 \in \mathbb{F}_q^\times \\ \kappa \in \pi_F \mathcal{O}_F/\pi_F^n \mathcal{O}_F}} \Bigg( \sum_{\omega \in \pi_F^{n-1}\mathcal{O}_F/\pi_F^n\mathcal{O}_F} \zeta(1+(s+1-\gamma[\lambda_1])\omega)) \Bigg) [\lambda_1]^i \zeta(1+\gamma\kappa) \begin{pmatrix} [\lambda_1]+\kappa & 1 \\ 1 & 0 \end{pmatrix} \phi_{I_n}$$

This is equal to:

$$0 \text{ if } \gamma = 0 \bmod \pi_F \text{ and } s \neq -1 \bmod \pi_F,$$

$$qN_{i,\gamma} \text{ if } \gamma = 0 \bmod \pi_F \text{ and } s = -1 \bmod \pi_F,$$

$$0 \text{ if } \gamma \neq 0 \bmod \pi_F \text{ and } s = -1 \bmod \pi_F \text{ and}$$

$$q\overline{[\frac{s+1}{\gamma}]}^i \begin{pmatrix} \overline{[\frac{s+1}{\gamma}]} & 1 \\ 1 & 0 \end{pmatrix} N_\gamma \text{ if } \gamma \neq 0 \bmod \pi_F \text{ and } s \neq -1 \bmod \pi_F.$$

Let us now compute the action on $N_\gamma$. We have:

$$\begin{aligned}
\mathbb{O}_s^D N_\gamma &= \sum_{\substack{\lambda \in \pi_F \mathcal{O}_F/\pi_F^n \mathcal{O}_F \\ \omega \in \pi_F^{n-1}\mathcal{O}_F/\pi_F^n\mathcal{O}_F}} \zeta(1+s\omega)\chi_1^{-1}(1+\omega)\zeta(1+\gamma\lambda) \begin{pmatrix} 1+\omega & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \phi_{I_n} \\
&= \sum_{\substack{\lambda \in \pi_F \mathcal{O}_F/\pi_F^n \mathcal{O}_F \\ \omega \in \pi_F^{n-1}\mathcal{O}_F/\pi_F^n\mathcal{O}_F}} \zeta(1+s\omega)\chi_1^{-1}(1+\omega)\zeta(1+\gamma\lambda) \begin{pmatrix} 1 & 0 \\ \frac{\lambda}{1+\omega} & 1 \end{pmatrix} \begin{pmatrix} 1+\omega & 0 \\ 0 & 1 \end{pmatrix} \phi_{I_n}.
\end{aligned}$$

Let us define $\kappa = \frac{\lambda}{1+\omega}$. Using that $\kappa\omega = 0 \bmod \pi_F^n$ we continue with the computation:

$$\begin{aligned}
\mathbb{O}_s^D N_\gamma &= \sum_{\substack{\kappa \in \pi_F \mathcal{O}_F/\pi_F^n \mathcal{O}_F \\ \omega \in \pi_F^{n-1}\mathcal{O}_F/\pi_F^n\mathcal{O}_F}} \zeta(1+s\omega)\chi_1^{-1}(1+\omega)\zeta(1+\gamma\kappa)\chi_1(1+\omega) \begin{pmatrix} 1 & 0 \\ \kappa & 1 \end{pmatrix} \phi_{I_n} \\
&= \sum_{\substack{\kappa \in \pi_F \mathcal{O}_F/\pi_F^n \mathcal{O}_F \\ \omega \in \pi_F^{n-1}\mathcal{O}_F/\pi_F^n\mathcal{O}_F}} \zeta(1+s\omega)\zeta(1+\gamma\kappa) \begin{pmatrix} 1 & 0 \\ \kappa & 1 \end{pmatrix} \phi_{I_n} \\
&= \sum_{\kappa \in \pi_F \mathcal{O}_F/\pi_F^n \mathcal{O}_F} \left( \sum_{\omega \in \pi_F^{n-1}\mathcal{O}_F/\pi_F^n\mathcal{O}_F} \zeta(1+s\omega) \right) \zeta(1+\gamma\kappa) \begin{pmatrix} 1 & 0 \\ \kappa & 1 \end{pmatrix} \phi_{I_n}
\end{aligned}$$

This is equal to $qN_\gamma$ if $s=0$ and 0 otherwise. Let us now compute the action on $\mathbb{O}_s^D N^\gamma$. We have:

$$\begin{aligned}
\mathbb{O}_s^D N^\gamma &= \sum_{\substack{\lambda\in\pi_F\mathcal{O}_F/\pi_F^n\mathcal{O}_F \\ \omega\in\pi_F^{n-1}\mathcal{O}_F/\pi_F^n\mathcal{O}_F}} \zeta(1+s\omega)\chi_1^{-1}(1+\omega)\zeta(1+\gamma\lambda)\begin{pmatrix} 1+\omega & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} \lambda & 1 \\ 1 & 0\end{pmatrix}\phi_{I_n} \\
&= \sum_{\substack{\lambda\in\pi_F\mathcal{O}_F/\pi_F^n\mathcal{O}_F \\ \omega\in\pi_F^{n-1}\mathcal{O}_F/\pi_F^n\mathcal{O}_F}} \zeta(1+s\omega)\chi_1^{-1}(1+\omega)\zeta(1+\gamma\lambda)\begin{pmatrix} \lambda(1+\omega) & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1+\omega\end{pmatrix}\phi_{I_n}.
\end{aligned}$$

Define $\kappa = \lambda(1+\omega)$. We continue with the computation:

$$\begin{aligned}
\mathbb{O}_s^D N^\gamma &= \sum_{\substack{\kappa\in\pi_F\mathcal{O}_F/\pi_F^n\mathcal{O}_F \\ \omega\in\pi_F^{n-1}\mathcal{O}_F/\pi_F^n\mathcal{O}_F}} \zeta(1+s\omega)\chi_1^{-1}\chi_2(1+\omega)\zeta(1+\gamma\frac{\kappa}{1+\omega})\begin{pmatrix} \kappa & 1 \\ 1 & 0\end{pmatrix}\phi_{I_n} \\
&= \sum_{\substack{\kappa\in\pi_F\mathcal{O}_F/\pi_F^n\mathcal{O}_F \\ \omega\in\pi_F^{n-1}\mathcal{O}_F/\pi_F^n\mathcal{O}_F}} \zeta(1+s\omega)\zeta(1+\omega)\zeta^{-1}(1+\omega)\zeta(1+\omega+\gamma\kappa)\begin{pmatrix} \kappa & 1 \\ 1 & 0\end{pmatrix}\phi_{I_n} \\
&= \sum_{\kappa\in\pi_F\mathcal{O}_F/\pi_F^n\mathcal{O}_F}\Bigg(\sum_{\omega\in\pi_F^{n-1}\mathcal{O}_F/\pi_F^n\mathcal{O}_F}\zeta(1+\omega(s+1))\Bigg)\zeta(1+\gamma\kappa)\begin{pmatrix} \kappa & 1 \\ 1 & 0\end{pmatrix}\phi_{I_n} \\
&= \Bigg(\sum_{\omega\in\pi_F^{n-1}\mathcal{O}_F/\pi_F^n\mathcal{O}_F}\zeta(1+\omega(s+1))\Bigg)\sum_{\kappa\in\pi_F\mathcal{O}_F/\pi_F^n\mathcal{O}_F}\zeta(1+\gamma\kappa)\begin{pmatrix} \kappa & 1 \\ 1 & 0\end{pmatrix}\phi_{I_n} \\
&= \Bigg(\sum_{\omega\in\pi_F^{n-1}\mathcal{O}_F/\pi_F^n\mathcal{O}_F}\zeta(1+\omega(s+1))\Bigg)N^\gamma
\end{aligned}$$

This is equal to $qN^\gamma$ when $s=-1 \bmod \pi_F$ and 0 otherwise.

Let us define

$$\mathbb{P}_s^t := \sum_{\omega\in\pi_F^t\mathcal{O}_F/\pi_F^{t+1}\mathcal{O}_F}\zeta(1+s\omega)\chi_2^{-1}(1+s\omega)\begin{pmatrix} 1 & 0 \\ \omega & 1\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1+s\omega\end{pmatrix},$$

for $s \in \mathcal{O}_F/\pi_F^{n-1}$, $1 \leq t \leq n-2$.

We will show that for $s = \gamma \bmod \pi_F^{n-t-1}$ we have a well defined action of $\mathbb{P}_s^t$ on $N_\gamma$, that is we will how that $\mathbb{P}_s^t N_\gamma$ is independent of the choice of representatives of $\omega \bmod \pi_F^{t+1}$. For $s = \gamma \bmod \pi_F^{n-t-1}$ and $o_\omega \in \mathcal{O}_F$ we have

$$\begin{aligned}
\mathbb{P}_s^t N_\gamma = & \sum_{\omega \in \pi_F^t \mathcal{O}_F/\pi_F^{t+1}\mathcal{O}_F} \zeta(1+s(\omega+o_\omega\pi_F^{t+1}))\chi_2^{-1}(1+s(\omega+o_\omega\pi_F^{t+1})) \\
& \begin{pmatrix} 1 & 0 \\ \omega+o_\omega\pi_F^{t+1} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1+s(\omega+o_\omega\pi_F^{t+1}) \end{pmatrix} N_\gamma \\
= & \sum_{\substack{\omega \in \pi_F^t \mathcal{O}_F/\pi_F^{t+1}\mathcal{O}_F \\ \lambda \in \pi_F \mathcal{O}_F/\pi_F^n \mathcal{O}_F}} \zeta(1+s(\omega+o_\omega\pi_F^{t+1}))\chi_2^{-1}(1+s(\omega+o_\omega\pi_F^{t+1}))\zeta(1+\gamma\lambda) \\
& \begin{pmatrix} 1 & 0 \\ \omega+o_\omega\pi_F^{t+1} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1+s(\omega+o_\omega\pi_F^{t+1}) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \phi_{I_n} \\
= & \sum_{\substack{\omega \in \pi_F^t \mathcal{O}_F/\pi_F^{t+1}\mathcal{O}_F \\ \lambda \in \pi_F \mathcal{O}_F/\pi_F^n \mathcal{O}_F}} \zeta(1+s(\omega+o_\omega\pi_F^{t+1}))\chi_2^{-1}(1+s(\omega+o_\omega\pi_F^{t+1}))\zeta(1+\gamma\lambda) \\
& \begin{pmatrix} 1 & 0 \\ \omega+o_\omega\pi_F^{t+1} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \lambda(1+s(\omega+o_\omega\pi_F^{t+1})) & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1+s(\omega+o_\omega\pi_F^{t+1}) \end{pmatrix} \phi_{I_n} \\
= & \sum_{\substack{\omega \in \pi_F^t \mathcal{O}_F/\pi_F^{t+1}\mathcal{O}_F \\ \lambda \in \pi_F \mathcal{O}_F/\pi_F^n \mathcal{O}_F}} \zeta(1+s(\omega+o_\omega\pi_F^{t+1}))\zeta(1+\gamma\lambda) \begin{pmatrix} 1 & 0 \\ \omega+o_\omega\pi_F^{t+1}+\lambda(1+s(\omega+o_\omega\pi_F^{t+1})) & 1 \end{pmatrix} \phi_{I_n}
\end{aligned}$$

Let us define $\kappa := \omega+o_\omega\pi_F^{t+1}+\lambda(1+s(\omega+o_\omega\pi_F^{t+1}))$. We continue with the computation:

$$\begin{aligned}
\mathbb{P}_s^t N_\gamma &= \sum_{\substack{\omega\in\pi_F^t\mathcal{O}_F/\pi_F^{t+1}\mathcal{O}_F\\ \kappa\in\pi_F\mathcal{O}_F/\pi_F^n\mathcal{O}_F}} \zeta(1+s(\omega+o_\omega\pi_F^{t+1}))\zeta(1+\gamma\frac{\kappa-(\omega+o_\omega\pi_F^{t+1})}{1+s(\omega+o_\omega\pi_F^{t+1})}) \begin{pmatrix} 1 & 0 \\ \kappa & 1 \end{pmatrix} \phi_{I_n} \\
&= \sum_{\substack{\omega\in\pi_F^t\mathcal{O}_F/\pi_F^{t+1}\mathcal{O}_F\\ \kappa\in\pi_F\mathcal{O}_F/\pi_F^n\mathcal{O}_F}} \zeta(1+s(\omega+o_\omega\pi_F^{t+1})+\gamma(\kappa-\omega-o_\omega\pi_F^{t-1})) \begin{pmatrix} 1 & 0 \\ \kappa & 1 \end{pmatrix} \phi_{I_n} \\
&= \sum_{\substack{\omega\in\pi_F^t\mathcal{O}_F/\pi_F^{t+1}\mathcal{O}_F\\ \kappa\in\pi_F\mathcal{O}_F/\pi_F^n\mathcal{O}_F}} \zeta(1+(s-\gamma)(\omega+o_\omega\pi_F^{t+1})+\gamma\kappa) \begin{pmatrix} 1 & 0 \\ \kappa & 1 \end{pmatrix} \phi_{I_n} \\
&= \sum_{\substack{\omega\in\pi_F^t\mathcal{O}_F/\pi_F^{t+1}\mathcal{O}_F\\ \kappa\in\pi_F\mathcal{O}_F/\pi_F^n\mathcal{O}_F}} \zeta(1+(s-\gamma)\omega+\gamma\kappa) \begin{pmatrix} 1 & 0 \\ \kappa & 1 \end{pmatrix} \phi_{I_n}
\end{aligned}$$

where the last equality follows from the fact that $(s-\gamma)o_\omega\pi_F^{t+1} = 0 \bmod \pi_F^n$. We continue with the computation:

$$\begin{aligned}
\mathbb{P}_s^t N_\gamma &= \sum_{\substack{\omega\in\pi_F^t\mathcal{O}_F/\pi_F^{t+1}\mathcal{O}_F\\ \kappa\in\pi_F\mathcal{O}_F/\pi_F^n\mathcal{O}_F}} \zeta(1+(s-\gamma)\omega)\zeta(1+\gamma\kappa) \begin{pmatrix} 1 & 0 \\ \kappa & 1 \end{pmatrix} \phi_{I_n} \\
&= \sum_{\kappa\in\pi_F\mathcal{O}_F/\pi_F^n\mathcal{O}_F} \big( \sum_{\omega\in\pi_F^t\mathcal{O}_F/\pi_F^{t+1}\mathcal{O}_F} \zeta(1+(s-\gamma)\omega)\big)\zeta(1+\gamma\kappa) \begin{pmatrix} 1 & 0 \\ \kappa & 1 \end{pmatrix} \phi_{I_n} \\
&= \big( \sum_{\omega\in\pi_F^t\mathcal{O}_F/\pi_F^{t+1}\mathcal{O}_F} \zeta(1+(s-\gamma)\omega)\big) \sum_{\kappa\in\pi_F\mathcal{O}_F/\pi_F^n\mathcal{O}_F} \zeta(1+\gamma\kappa) \begin{pmatrix} 1 & 0 \\ \kappa & 1 \end{pmatrix} \phi_{I_n} \\
&= \big( \sum_{\omega\in\pi_F^t\mathcal{O}_F/\pi_F^{t+1}\mathcal{O}_F} \zeta(1+(s-\gamma)\omega)\big) N_\gamma.
\end{aligned}$$

Since $\zeta$ is non trivial on $1+\pi_F^{n-1}\mathcal{O}_F$, we have that $\mathbb{P}_s N_\gamma = 0$ if $s-\gamma \neq 0 \bmod \pi_F^{n-t}$.

That is $\mathbb{P}_s N_\gamma = 0$ if $s - \gamma \neq 0$. If $s = \gamma \bmod \pi_F^{n-t}$ we have that

$$\mathbb{P}_s^t N_\gamma = q N_\gamma.$$

In particular, we have that $\mathbb{P}_s := \mathbb{P}_s^1 \mathbb{P}_s^2 ... \mathbb{P}_s^{n-2}$ has a well defined action on $N_\gamma$, for $s = \gamma \bmod \pi_F$. For $s = \gamma \bmod \pi_F$ we have that $P_s N_\gamma = q^{n-2} N_\gamma$, if $s = \gamma \bmod \pi_F^{n-1}$ and $P_s N_\gamma = 0$ otherwise.

**Theorem 3.2.2.** *Let $\Lambda \subset \Pi_n$ be $G$ stable lattice such that $\Lambda \not\subset \pi_E \Pi_n$. Then we have that $q^n \Pi_n \subset \Lambda$.*

*Proof.* Let $v = \sum_{\substack{0 \leq j < q-1 \\ \gamma \in \mathcal{O}_F/\pi_F^{n-1}\mathcal{O}_F}} a_{j,\gamma} N_{j,\gamma} + \sum_{\gamma \in \mathcal{O}_F/\pi_F^{n-1}\mathcal{O}_F} a_\gamma N_\gamma + \sum_{\gamma \in \mathcal{O}_F/\pi_F^{n-1}\mathcal{O}_F} a^\gamma N^\gamma \in \Lambda$ and $v \notin \pi_E \Pi_n$. We have a few cases.

Let us first assume that there is $a_{i,\beta}$ such that $a_{i,\beta} \in \mathcal{O}_E^\times$. Let us first consider the case when there is $a_{i,\beta} \in \mathcal{O}_E^\times$, such that $\beta \in \mathcal{O}_F^\times$. We then have

$$\mathbb{O}_\beta^T v = \sum_{\substack{0 \leq j < q-1 \\ \gamma \in \mathcal{O}_F/\pi_F^{n-1}\mathcal{O}_F \\ \gamma = \beta \bmod \pi_F}} q a_{j,\gamma} N_{j,\gamma} + q \sum_{\substack{\gamma \in \mathcal{O}_F/\pi_F^{n-1}\mathcal{O}_F \\ \gamma = \beta \bmod \pi_F}} a^\gamma N^\gamma \in \Lambda.$$

For $s \neq -1 \bmod \pi_F$ we have

$$\mathbb{O}_s^D \mathbb{O}_\beta^T v = \sum_{\substack{0 \leq j < q-1 \\ \gamma \in \mathcal{O}_F/\pi_F^{n-1}\mathcal{O}_F \\ \gamma = \beta \bmod \pi_F}} q a_{j,\gamma} \sum_{\kappa \in \pi_F \mathcal{O}_F/\pi_F^n \mathcal{O}_F} q[\frac{\bar{s}+1}{\bar{\beta}}]^j \zeta(1+\gamma\kappa) \begin{pmatrix} [\frac{\bar{s}+1}{\bar{\beta}}] + \kappa & 1 \\ 1 & 0 \end{pmatrix} \phi_{I_n} \in \Lambda.$$

Notice that

$$\begin{pmatrix} [\lambda] + \kappa & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} [\lambda] & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \kappa & 1 \end{pmatrix},$$

so we have

$$\begin{pmatrix} [\frac{\bar{s}+1}{\bar{\beta}}] & 1 \\ 1 & 0 \end{pmatrix}^{-1} \mathbb{O}_s^D \mathbb{O}_\beta^T v = \sum_{\substack{0\le j<q-1 \\ \gamma\in\mathcal{O}_F/\pi_F^{n-1}\mathcal{O}_F \\ \gamma=\beta \bmod \pi_F}} q^2 a_{j,\gamma} \sum_{\kappa\in\pi_F\mathcal{O}_F/\pi_F^n\mathcal{O}_F} [\frac{\bar{s}+1}{\bar{\beta}}]^j \zeta(1+\gamma\kappa) \begin{pmatrix} 1 & 0 \\ \kappa & 1 \end{pmatrix} \phi_{I_n}$$

$$= \sum_{\substack{0\le j<q-1 \\ \gamma\in\mathcal{O}_F/\pi_F^{n-1}\mathcal{O}_F \\ \gamma=\beta \bmod \pi_F}} q^2 a_{j,\gamma} [\frac{\bar{s}+1}{\bar{\beta}}]^j \sum_{\kappa\in\pi_F\mathcal{O}_F/\pi_F^n\mathcal{O}_F} \zeta(1+\gamma\kappa) \begin{pmatrix} 1 & 0 \\ \kappa & 1 \end{pmatrix} \phi_{I_n}$$

$$= \sum_{\substack{0\le j<q-1 \\ \gamma\in\mathcal{O}_F/\pi_F^{n-1}\mathcal{O}_F \\ \gamma=\beta \bmod \pi_F}} q^2 a_{j,\gamma} [\frac{\bar{s}+1}{\bar{\beta}}]^j N_\gamma = q^2 \sum_{\substack{\gamma\in\mathcal{O}_F/\pi_F^{n-1}\mathcal{O}_F \\ \gamma=\beta \bmod \pi_F}} \Big( \sum_{0\le j<q-1} a_{j,\gamma} [\frac{\bar{s}+1}{\bar{\beta}}]^j \Big) N_\gamma$$

$$\in \Lambda.$$

That is for any $\lambda \in \mathbb{F}_q^\times$ we have

$$\begin{pmatrix} [\lambda] & 1 \\ 1 & 0 \end{pmatrix}^{-1} \mathbb{O}_{[\lambda]\beta-1}^D \mathbb{O}_\beta^T v = q^2 \sum_{\substack{\gamma\in\mathcal{O}_F/\pi_F^{n-1}\mathcal{O}_F \\ \gamma=\beta \bmod \pi_F}} \Big( \sum_{0\le j<q-1} [\lambda]^j a_{j,\gamma} \Big) N_\gamma \in \Lambda.$$

Since $\Big([\lambda]^j\Big)_{\substack{\lambda\in\mathbb{F}_q^\times \\ 0\le j<q-1}} \in \mathrm{GL}_{q-1}(\mathcal{O}_E)$ and since $a_{i,\beta} \in \mathcal{O}_E^\times$ we have that there is $\lambda_\beta \in \mathbb{F}_q^\times$ such that $\sum_{0\le j<q-1} [\lambda_\beta]^j a_{j,\beta} \in \mathcal{O}_F^\times$. So we have

$$w := q^2 \sum_{\substack{\gamma\in\mathcal{O}_F/\pi_F^{n-1}\mathcal{O}_F \\ \gamma=\beta \bmod \pi_F}} x_\gamma N_\gamma \in \Lambda,$$

where $x_\gamma := \sum_{0\le j<q-1} [\lambda_\beta]^j a_{j,\gamma}$, $x_\gamma \in \mathcal{O}_F$ and $x_\beta \in \mathcal{O}_F^\times$. Notice that $\mathbb{P}_\beta$ has a well defined action on $w$, according to our previous discussion. In fact, we have

$$\mathbb{P}_\beta w = q^2 x_\beta q^{n-2} N_\beta \in \Lambda,$$

which gives

$$\mathbb{P}_\beta w = q^n x_\beta N_\beta \in \Lambda.$$

That is we have that $q^n N_\beta \in \Lambda$. But this gives that $q^n \Pi_n = q^n \mathcal{O}_E[G] N_\beta \subset \Lambda$.

Let us now consider the case when $\beta = 0 \bmod \pi_F$, $a_{j,\beta} \in \mathcal{O}_E^\times$ and $a_{j,\gamma} \notin \mathcal{O}_E^\times$ for $\gamma \neq 0 \bmod \pi_F$, $0 \leq j < q-1$. Recall that

$$\Pi_n = \bigoplus_{\substack{\lambda \in \mathbb{F}_q^\times \\ \gamma \in \mathcal{O}_F/\pi_F^{n-1}\mathcal{O}_F}} \mathcal{O}_E \begin{pmatrix} [\lambda] & 1 \\ 1 & 0 \end{pmatrix} N_\gamma \bigoplus_{\gamma \in \mathcal{O}_F/\pi_F^{n-1}\mathcal{O}_F} \mathcal{O}_E N_\gamma + \bigoplus_{\gamma \in \mathcal{O}_F/\pi_F^{n-1}\mathcal{O}_F} \mathcal{O}_E N^\gamma$$

and that

$$\bigoplus_{\lambda \in \mathbb{F}_q^\times} \mathcal{O}_E \begin{pmatrix} [\lambda] & 1 \\ 1 & 0 \end{pmatrix} N_\gamma = \bigoplus_{0 \leq j < q-1} \mathcal{O}_E N_{j,\gamma}.$$

We have

$$\begin{aligned}
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} N_{i,\beta} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sum_{\lambda_1 \in \mathbb{F}_q^\times} [\lambda_1]^i \begin{pmatrix} [\lambda_1] & 1 \\ 1 & 0 \end{pmatrix} N_\beta \\
&= \sum_{\lambda_1 \in \mathbb{F}_q^\times} [\lambda_1]^i \begin{pmatrix} [\lambda_1^{-1}] & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} [\lambda_1] & 0 \\ 0 & -[\lambda_1^{-1}] \end{pmatrix} \begin{pmatrix} 1 & [\lambda_1^{-1}] \\ 0 & 1 \end{pmatrix} N_\beta \\
&= \sum_{\lambda_1 \in \mathbb{F}_q^\times} [\lambda_1]^i \begin{pmatrix} [\lambda_1^{-1}] & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} [\lambda_1] & 0 \\ 0 & -[\lambda_1^{-1}] \end{pmatrix} N_{\beta - [\lambda_1^{-1}]} \\
&= \chi_2(-1) \sum_{\lambda_1 \in \mathbb{F}_q^\times} [\lambda_1]^i \chi_1 \chi_2^{-1}([\lambda_1]) \begin{pmatrix} [\lambda_1^{-1}] & 1 \\ 1 & 0 \end{pmatrix} N_{(\beta - [\lambda_1^{-1}]) \frac{[\lambda_1]}{(-[\lambda_1^{-1}])}} \\
&= (-1)^{c_2} \sum_{\lambda_1 \in \mathbb{F}_q^\times} [\lambda_1]^{i+c} \begin{pmatrix} [\lambda_1^{-1}] & 1 \\ 1 & 0 \end{pmatrix} N_{[\lambda_1] - \beta[\lambda_1]^2} \\
&= (-1)^{c_2} \sum_{\lambda_1 \in \mathbb{F}_q^\times} [\lambda_1]^{-i-c} \begin{pmatrix} [\lambda_1] & 1 \\ 1 & 0 \end{pmatrix} N_{[\lambda_1^{-1}] - \beta[\lambda_1]^{-2}}.
\end{aligned}$$

Denote $F_{i,\gamma} := (-1)^{c_2} \sum_{\lambda_1 \in \mathbb{F}_q^\times} [\lambda_1]^{-i-c} \begin{pmatrix} [\lambda_1] & 1 \\ 1 & 0 \end{pmatrix} N_{[\lambda_1^{-1}] - \gamma[\lambda_1]^{-2}}$ for $\gamma \in \pi_F \mathcal{O}_F / \pi_F^{n-1} \mathcal{O}_F$

and $0 \leq i < q-1$. Let us assume that $[\lambda_1^{-1}] - \beta_1[\lambda_1]^{-2} = [\lambda_2^{-1}] - \beta_2[\lambda_2]^{-2}$. Then reducing modulo $\pi_F$ on both sides, we see that we must have $\lambda_1 = \lambda_2$. This implies that we must have $\beta_1 = \beta_2$ as well. This plus having on mind that the matrix $([\lambda]^{-i-c})_{\substack{\lambda\in\mathbb{F}_q^\times \\ 0\leq i<q-1}} \in \mathrm{GL}_2(\mathcal{O}_F)$ and $([\lambda]^{i})_{\substack{\lambda\in\mathbb{F}_q^\times \\ 0\leq i<q-1}} \in \mathrm{GL}_2(\mathcal{O}_F)$ gives us that the vector space $A := \bigoplus_{\substack{0\leq i<q-1 \\ \gamma\in\mathcal{O}_F^\times}} F_{i,\gamma} = \bigoplus_{\substack{\lambda\in\mathbb{F}_q^\times \\ \gamma\in\mathcal{O}_F^\times}} \begin{pmatrix} [\lambda] & 1 \\ 1 & 0 \end{pmatrix} N_\gamma = \bigoplus_{\substack{0\leq i<q-1 \\ \gamma\in\mathcal{O}_F^\times}} N_{i,\gamma}$. This implies that even if there are some other $\beta_1, ..., \beta_k \in \pi_F\mathcal{O}_F$ and $0 \leq i_1, ..., i_k < q-1$ with the corresponding $\alpha_{i_1,\beta_1}, ..., \alpha_{i_k,\beta_k} \in \mathcal{O}_F^\times$ since $\alpha_{i,\beta}N_{i,\beta} + \alpha_{i_1,\beta_1}N_{i_1,\beta_1} + ... + \alpha_{i_k,\beta_k}N_{i_k,\beta_k}$ is non zero in $\Pi_n/\pi_F\Pi_n$ then $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(\alpha_{i,\beta}N_{i,\beta} + \alpha_{i_1,\beta_1}N_{i_1,\beta_1} + ... + \alpha_{i_k,\beta_k}N_{i_k,\beta_k})$ is non zero in $\Pi_n/\pi_F\Pi_n$ as well. But since $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(\alpha_{i,\beta}N_{i,\beta} + \alpha_{i_1,\beta_1}N_{i_1,\beta_1} + ... + \alpha_{i_k,\beta_k}N_{i_k,\beta_k}) \in A$, we have that $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(\alpha_{i,\beta}N_{i,\beta} + \alpha_{i_1,\beta_1}N_{i_1,\beta_1} + ... + \alpha_{i_k,\beta_k}N_{i_k,\beta_k})$ is non zero in $A/\pi_F A$. But since $A = \bigoplus_{\substack{0\leq i<q-1 \\ \gamma\in\mathcal{O}_F^\times}} N_{i,\gamma}$ this implies that $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} v$ is non zero in

$$\Pi_n/(\bigoplus_{\substack{0\leq i<q-1 \\ \gamma\in\pi_F\mathcal{O}_F/\pi_F^{n-1}\mathcal{O}_F}} \mathcal{O}_E N_{i,\gamma} \bigoplus_{\gamma\in\mathcal{O}_F/\pi_F^{n-1}\mathcal{O}_F} \mathcal{O}_E N_\gamma \bigoplus_{\gamma\in\mathcal{O}_F/\pi_F^{n-1}\mathcal{O}_F} \mathcal{O}_E N^\gamma, \pi_E\Pi_n).$$

So we have that $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} v$ is as in the first case.

Let us now consider the case when there is $a_\beta \in \mathcal{O}_E^\times$ and $a_{j,\gamma} \notin \mathcal{O}_E^\times$ for $\gamma \in \mathcal{O}_F/\pi_F^{n-1}\mathcal{O}_F$, $0 \leq j < q-1$. We have that

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} v \in \bigoplus_{\substack{\lambda\in\mathbb{F}_q^\times \\ \gamma\in\mathcal{O}_F^\times}} \begin{pmatrix} [\lambda] & 1 \\ 1 & 0 \end{pmatrix} N_\gamma = \bigoplus_{\substack{0\leq i<q-1 \\ \gamma\in\mathcal{O}_F^\times}} N_{i,\gamma}$$

That is we have that

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} v \in \Pi_n/(\bigoplus_{\gamma \in \mathcal{O}_F/\pi_F^{n-1}\mathcal{O}_F} \mathcal{O}_E N_\gamma \bigoplus_{\gamma \in \mathcal{O}_F/\pi_F^{n-1}\mathcal{O}_F} \mathcal{O}_E N^\gamma, \pi_E \Pi_n)$$

is non-zero, which reduces this case to the previous cases.

Notice that for the final case when there is $a^\beta \in \mathcal{O}_E^\times$ and $a_{j,\gamma}, a_\gamma \notin \mathcal{O}_E^\times$ for $\gamma \in \mathcal{O}_F/\pi_F^{n-1}\mathcal{O}_F$, $0 \leq j < q-1$, we have that $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} v$ is as in the previous case. We have thus exhausted all cases and our proof concludes. $\square$

# Chapter 4

# Computational methods

In this section we describe how we computed lattices in different cases in Magma. The code for the main example, described in 3.2, is shown in 4.3.

## 4.1 Explanation of computing all lattices in Magma

The computation of lattices for a specific example is computed in a similar way for all cases. Let $\rho : G \to V$ be a representation of a group $G$ over an $E-$ vector space $V$, where $E$ is a $p$ - adic field with ring of integers $\mathcal{O}_E$. We also assume that $G/ker(\rho)$ is finite. We fix a basis $e_1, ..., e_n$ of $V$ and a $\mathcal{O}_E[G]$ - stable latice $\Lambda \subset V$, as defined in 2.2. Each lattice in the code we represent as a matrix with respect to a fixed basis $e_1, ..., e_n$, where the basis vectors of the lattice are the columns of the corresponding matrix. We are heavily using the Magma function $Submodules()$ which is computing submodules over a finite field. Starting with the fixed lattice $\Lambda$, we compute all $k_E[G]$ submodules $\bar{\Lambda}_i$ of the $k_E[G]$ module $\bar{\Lambda} := \Lambda/\pi_E\Lambda$ such that $\bar{\Lambda}/\bar{\Lambda}_i$ is irreducible. On this way once when we iterate over submodules of $\bar{\Lambda}$ we compute all outgoing edges from $\Lambda$. In the code we actually implemented that when we iterate over all $k_E[G]$ submodules $\bar{\Lambda}_i$ of the $k_E[G]$ module $\bar{\Lambda} := \Lambda/\pi_E\Lambda$ and we check if $\bar{\Lambda}/\bar{\Lambda}_i$ or $\bar{\Lambda}_i$ is irreducible. On this way once when we iterate over submodules of $\bar{\Lambda}$ we compute all directed edges going from and out of $\Lambda$. In this way we compute all directed neighbours of $\Lambda$. We implemented it like this since when we work with big graphs in which case running the code takes very long time, we know that we computed all directed edges from and out of lattices for which we completed the iteration above. We apply the same process on newly found lattices and iterate until we find all the lattices. We know that the algorithm must finish, since there are a finite number of lattices, as shown in 2.2.4. In 4.3 the full code for computing lattices in $ind_{B_2}^{G}\chi$ is shown, for $p = 3$, $F = \mathbb{Q}_3$ and $E = \mathbb{Q}_3[\zeta_3]$, $\chi_2 = 1$, $\chi_1$ is such that sends $-1$ to $-1$ and 4 to $\zeta_3$.

## 4.2 Results for $p = 2, 3, 5$

In this chapter we will describe in detail the code for computing lattices in a representation of wild principal series type of $\mathrm{GL}_2(\mathcal{O}_F)$ over E, where $F = \mathbb{Q}_p$ and $E = F[\zeta_p]$. The code is shown in 4.3.1. We will first describe the notation in 4.3.1. With `p` we denote a prime $p$, with `d` we denote the dimension of the representation, which in this case is $p(p+1)$. With `F` we denote the $F = \mathbb{Q}_p$ and with `O_F` we denote its ring of integers $\mathcal{O}_F$. With `f1` we denote a polynomial $f(x)$ over $F$ with splitting field $E$. We use this polynomial in order to construct the field $E$ in Magma, denoted by `E`. With `t` we denote the uniformizer of $E$, which is $\zeta_p - 1$ and with `zeta_p` we denote a non-trivial $p$ -th root of unity $\zeta_p$.

With `gen1` we denote a generator of Teichmuller lifts in $F$ and with `gen2` we denote $p + 1$. Notice that any element from $(\mathcal{O}_F/\pi_F^2\mathcal{O}_F)^\times$ can be represented as $\mathtt{gen1}^{\mathtt{i1}}\mathtt{gen2}^{i2}$ (that is modulo $\pi_F^2$) for some integers `i1`, `i2`. With `imgen1` we denote a generator of Teichmuller lifts, which together with `imgen2` (which is $\zeta_p$) spans $E$ over $F$. We use this in order to construct a function which will be able to compute values of any character $(\mathcal{O}_F/\pi_F^2\mathcal{O}_F)^\times \to E$, the function is denoted in the code by `chi`. The function `chi` is determined by the integers `c1, c2`, generators `gen1, gen2` of $(\mathcal{O}_F/\pi_F^2\mathcal{O}_F)^\times$ and generators `imgen1, imgen2` of $E$. It is sending `gen1` to $\texttt{imgen1}^{\texttt{c1}}$ and `gen2` to $\texttt{imgen2}^{\texttt{c2}}$. In the code we have that $\chi = \chi_1 \otimes \chi_2$ where $\chi_1$ and $\chi_2$ are determined by the integers `ch1c1`, `ch1c2`, `ch2c1` and `ch2c2`.

We generate the action matrices of the generators of $\mathrm{GL}_2(\mathcal{O}_F)$ in the following way. Notice that $g_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $g_2 := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $g_3 := \begin{pmatrix} g_p & 0 \\ 0 & 1 \end{pmatrix}$, where $g_p$ is a generator of the group $(\mathcal{O}_F/\pi_F^2\mathcal{O}_F)^\times$. In the code $g_p$ is denoted as `gp` and the action of a generator $g_i$ is computed in the function `action_generatori`.

In what follows we describe how we computed the action matrices of the generators. Firstly we fix a basis with respect to which all action matrices and stable lattices will be computed. The basis which we fix is $e_i := \begin{pmatrix} 1 & 0 \\ pi & 1 \end{pmatrix} \phi_{I_2}$ for $1 \leq i \leq p$ and $e_j := \begin{pmatrix} j-p & 1 \\ 1 & 0 \end{pmatrix} \phi_{I_2}$ for $p+1 \leq j \leq p^2+p$. For $g_1$ and $e_k$, where $1 \leq k \leq p$ we have

$$g_1 e_k = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ pk & 1 \end{pmatrix} \phi_{I_2} = \begin{pmatrix} pk & 1 \\ 1 & 0 \end{pmatrix} \phi_{I_2} = e_{p+pk}.$$

For $e_{p+\lambda^\times}$, where $p+1 \le p+\lambda^\times \le p+p^2$ and $p \nmid \lambda^\times$ we have

$$\begin{aligned}
g_1 e_{p+\lambda^\times} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda^\times & 1 \\ 1 & 0 \end{pmatrix} \phi_{I_2} \\
&= \begin{pmatrix} 1 & 0 \\ \lambda^\times & 1 \end{pmatrix} \phi_{I_2} \\
&= \begin{pmatrix} (\lambda^\times)^{-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda^\times & 1 \\ 0 & -(\lambda^\times)^{-1} \end{pmatrix} \phi_{I_2} \\
&= \begin{pmatrix} (\lambda^\times)^{-1} & 1 \\ 1 & 0 \end{pmatrix} \chi_1(\lambda^\times)\chi_2(-(\lambda^\times)^{-1})\phi_{I_2} \\
&= \chi_1(\lambda^\times)\chi_2(-(\lambda^\times)^{-1}) e_{p+(\lambda^\times)^{-1}}
\end{aligned}$$

and for $e_{p+pk}$, where $1 \le k \le p$ we have

$$g_1 e_{p+pk} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} pk & 1 \\ 1 & 0 \end{pmatrix} \phi_{I_2} = \begin{pmatrix} 1 & 0 \\ pk & 1 \end{pmatrix} \phi_{I_2} = e_k.$$

We used the above calculation to write down the function `action_generator1`, which computes the matrix of the action of $g_1$ with respect to $e_1, ...e_{p^2+p}$. For $g_2$ and $e_k$, where $1 \leq k \leq p$ we have

$$\begin{aligned}
g_2 e_k &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ pk & 1 \end{pmatrix} \phi_{I_2} \\
&= \begin{pmatrix} 1+pk & 1 \\ pk & 1 \end{pmatrix} \phi_{I_2} \\
&= \begin{pmatrix} 1 & 0 \\ pk & 1 \end{pmatrix} \begin{pmatrix} 1+pk & 1 \\ 0 & 1-pk \end{pmatrix} \phi_{I_2} \\
&= \begin{pmatrix} 1 & 0 \\ pk & 1 \end{pmatrix} \chi_1(1+pk)\chi_2(1-pk)\phi_{I_2} \\
&= \chi_1\chi_2^{-1}(1+pk)e_k
\end{aligned}$$

and for $e_{p+\lambda}$, where $p+1 \leq p+\lambda \leq p+p^2$ we have

$$g_2 e_{p+\lambda} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 1 & 0 \end{pmatrix} \phi_{I_2} = \begin{pmatrix} \lambda+1 & 1 \\ 1 & 0 \end{pmatrix} \phi_{I_2} = e_{p+\lambda+1}$$

where $e_{p+p^2+1}$ denotes $e_{p+1}$. We used the above calculation in order to compute the action of $g_2$ in `action_generator2`.

For $g_3$ and $e_k$, where $1 \le k \le p$ we have

$$\begin{aligned}
g_3 e_k &= \begin{pmatrix} g_p & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ pk & 1 \end{pmatrix} \phi_{I_2} \\
&= \begin{pmatrix} g_p & 0 \\ pk & 1 \end{pmatrix} \phi_{I_2} \\
&= \begin{pmatrix} 1 & 0 \\ pkg_p^{-1} & 1 \end{pmatrix} \begin{pmatrix} g_p & 0 \\ 0 & 1 \end{pmatrix} \phi_{I_2} \\
&= \chi_1(g_p) \begin{pmatrix} 1 & 0 \\ pkg_p^{-1} & 1 \end{pmatrix} \phi_{I_2} \\
&= \chi_1(g_p) e_{kg_p^{-1} \bmod p}
\end{aligned}$$

where by $e_0$ we mean $e_p$. For $e_{p+\lambda}$, where $p+1 \le p+\lambda \le p+p^2$ we have

$$\begin{aligned}
g_3 e_{p+\lambda} &= \begin{pmatrix} g_p & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 1 & 0 \end{pmatrix} \phi_{I_2} \\
&= \begin{pmatrix} g_p\lambda & g_p \\ 1 & 0 \end{pmatrix} \phi_{I_2} \\
&= \begin{pmatrix} g_p\lambda & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & g_p \end{pmatrix} \phi_{I_2} \\
&= \chi_2(g_p) \begin{pmatrix} g_p\lambda & 1 \\ 1 & 0 \end{pmatrix} \phi_{I_2} \\
&= \chi_2(g_p) e_{p+(g_p\lambda \bmod p^2)}
\end{aligned}$$

where by $(p^2 t \bmod p^2)$ we we mean $p^2$. We used the above calculation in order to compute the action of $g_3$ in `action_generator3`.

We are searching for lattices in the following way. We represent each lattice as a matrix, whose columns are basis vectors of the lattice, with respect to the basis $e_1, ..., e_{p^2+p}$. Notice that $\Lambda_0 := \bigoplus_{i=1}^{n} \mathcal{O}_E e_i$ is stable, so we start with $\Lambda_0$ which we represent as the identity matrix. Then using Magma's function `Submodules`, which is computing submodules over a finite field, we compute all submodules of $\bar{\Lambda}_0 := \Lambda_0/\pi_E\Lambda_0$. We are interested in a submodule $\bar{\Lambda}$ of $\bar{\Lambda}_0$ such that the quotient $\bar{\Lambda}_0/\bar{\Lambda}$ is irreducible. We check this using Magma's function `IsIrreducible`. For all $\bar{\Lambda}$ such that $\bar{\Lambda}_0/\bar{\Lambda}$ is irreducible, we compute its lift $\Lambda$. Then we continue applying the process on lifts $\Lambda's$ until we reach the point when there are no non exploited classes of lattices, that is when we arrive in the iteration when we do not get any new lattice. We know that any lattice class $[\Lambda']$ will be exploited since if with $\Lambda'$ we denote the representative such that $\Lambda' \subseteq \Lambda$ and $\Lambda' \not\subseteq \pi_E\Lambda$ we have that there are lattices $\Lambda_1, ..., \Lambda_n$ such that $\Lambda_0 := \Lambda' \subset \Lambda_1 \subset ... \subset \Lambda_n \subset \Lambda_{n+1} := \Lambda$ and $\Lambda_{i+1}/\Lambda_i$ is irreducible for all $0 \leq i \leq n$. The code is shown in 4.3.

In the figure 4.1 we show the graph associated to the family of lattices for $p = 2$.

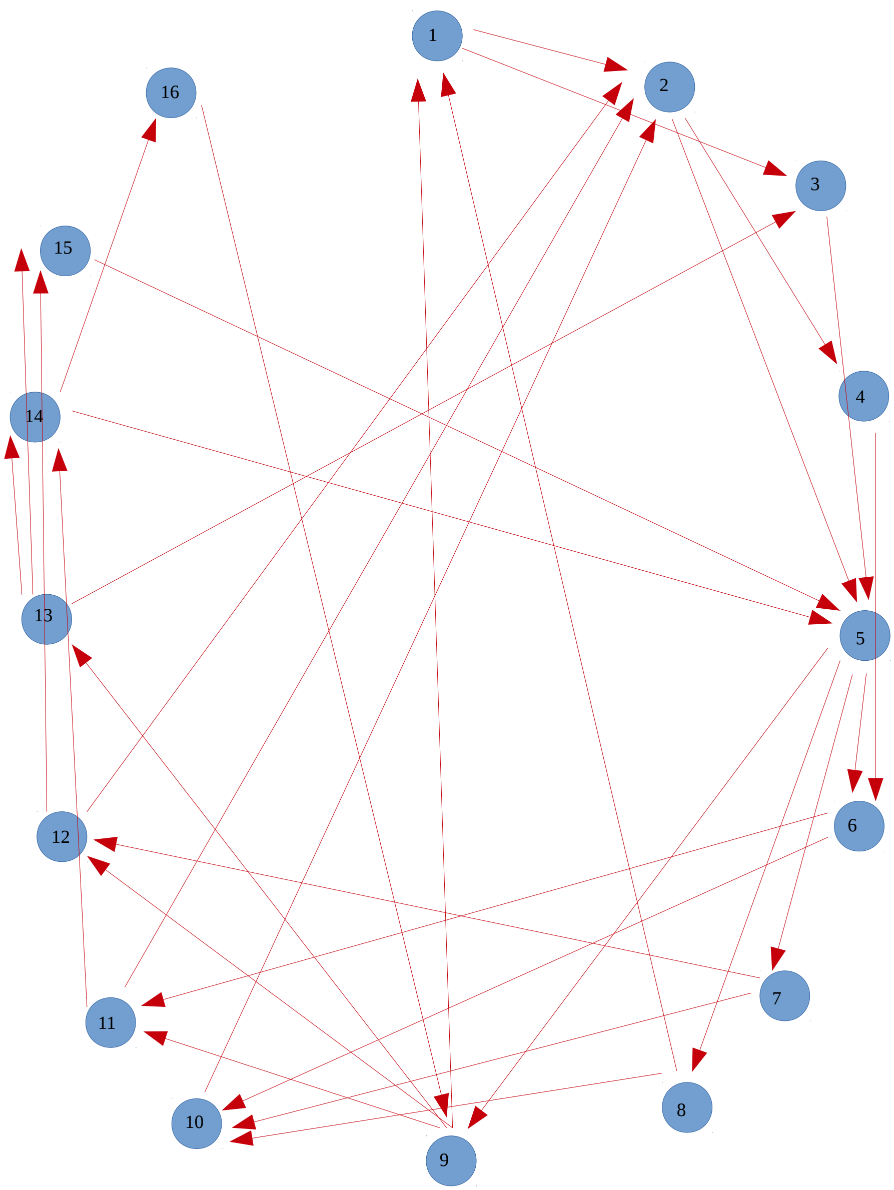


**Figure 4.1:** $p = 2$

The figure 4.2 shows the graph associated to the family of lattices for $p = 3$.

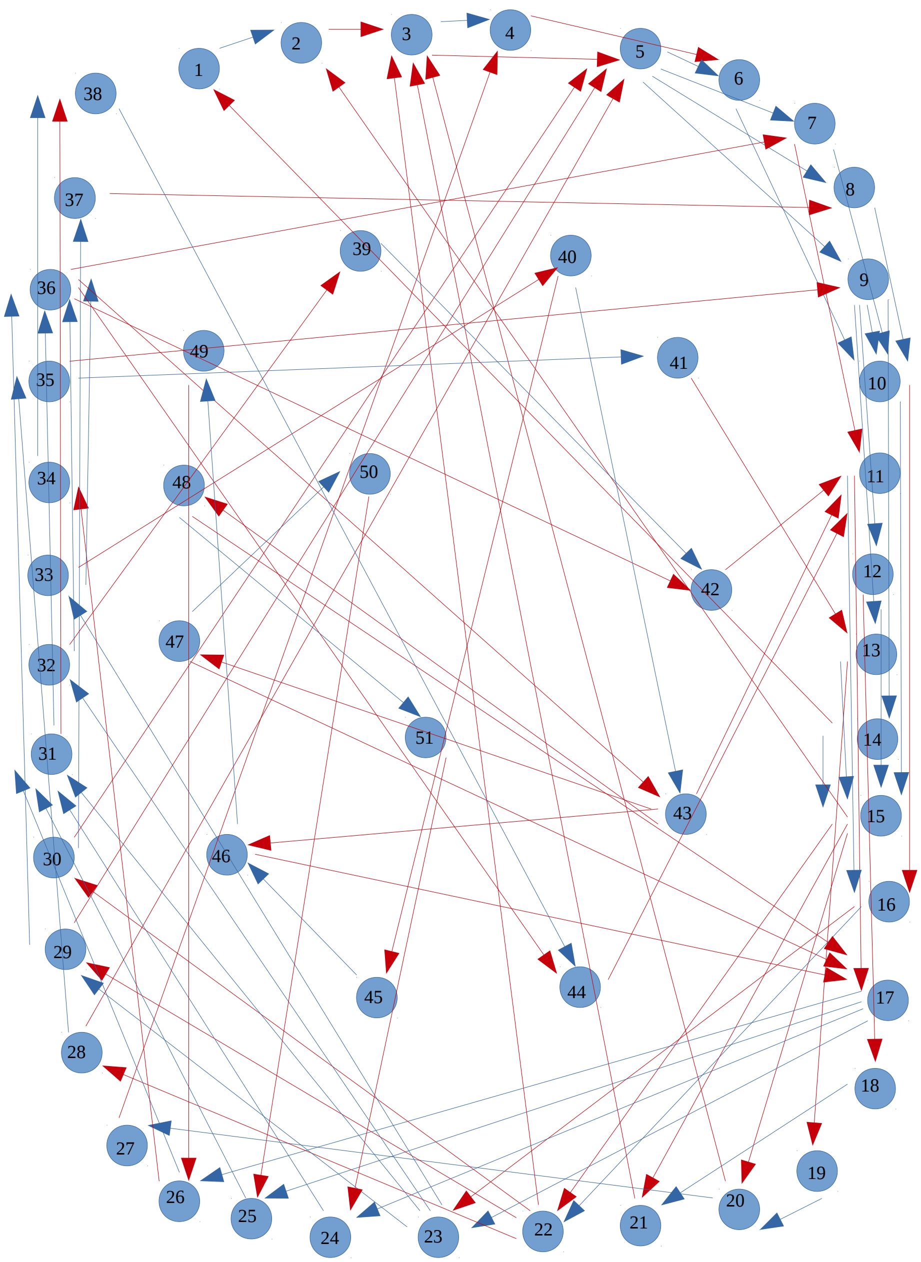
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**Figure 4.2:** $p = 3$

## 4.3 Code in Magma

### 4.3.1 Computation of stable lattices

In this section we present the full code for computing lattices in $ind_{B_2}^{G}\chi$. The setup is that $p = 3$, $F = \mathbb{Q}_3$ and $E = \mathbb{Q}_3[\zeta_3]$, $\chi_2 = 1$, $\chi_1$ is such that sends $-1$ to $-1$ and 4 to $\zeta_3$. The code is explained in 4.2.

```
p:=3;
d:= p*(p+1);
O_F := pAdicRing(p,4*p-1);
F:=pAdicField(p,4*p-1);
gen2:= F!(p+1);
ch1c1 :=1; // gen1 sent to gen1 ^c1
ch1c2 := 1; // 1+p sent to zeta_p ^c2
ch2c1:= 0;
ch2c2 := 0;
P1<x> := PolynomialRing(F);
f1 := x^2 +3*x +3;
E:= ext<F | f1 >;
k_E:=GF(p);
t:= E.1 ;
zeta_p := t + 1;
teich :=TeichmuellerSystem(O_F);
gen1 := teich[3];
imgen1 := E!gen1;
imgen2 := zeta_p;
gp := O_F!2;

lattice := ScalarMatrix(E,d,  1);

is_int := function(M)
    mini := Valuation(M[1][1]);
    for i in [1.. d] do
       for j in [1.. d] do
           mini := Min([mini, Valuation(E!M[i][j])]);
       end for;
    end for;


    if (mini ge 0) then
        return 1;
    end if;
    return 0;
end function;
```

```
chi := function(gen1, gen2, c1, c2, p, x)
    x := F!x;
    x1:= F!1;
    coef1 := 0;
    while Valuation(x-x1) lt 1 do
        x1:= x1*gen1;
        coef1 := coef1+1;
    end while;

    x2 := F!1;
    coef2 := 0;
    while  Valuation(x-(x1*x2)) lt 2 do
        x2:= x2*gen2;
        coef2 := coef2 +1;
    end while;

    to_return1 := imgen1^(coef1 * c1);
    to_return2 := imgen2^(coef2 * c2);
    to_return := to_return1*to_return2;
    return to_return;
end function;

inverse := function( p, x)
    p2 := p^2;
    for i in [1 .. p2] do
        if ((i*x) mod p2) eq 1 then
            return i;
        end if;
    end for;
end function;

action_generator1 := function(gen1, gen2, ch1c1, ch1c2, ch2c1, ch2c2, p)
    dim := p*(p+1);
    gR1 := ZeroMatrix(E,dim,dim);
    for k in [1 .. p] do
        gR1[(p*k)+p][k] := 1;
    end for;
    for l in [1 .. (p^2)] do
        if (l mod p) eq 0 then
            gR1[l div p][p+l] := 1;
        else
            lambda := l;
            part1 := chi(gen1, gen2, ch1c1, ch1c2, p, l);
            temp := (p^2)-(inverse(p, l));
            part2 :=chi(gen1, gen2, ch2c1, ch2c2, p, temp);
            gR1[p+inverse(p, l)][p+l] := (part1*part2) ;

        end if;
    end for;
    return gR1;
```

```
end function;

action_generator2 := function(gen1, gen2, ch1c1, ch1c2, ch2c1, ch2c2, p)
    dim := p*(p+1);
    gR1 := ZeroMatrix(E,dim,dim);
    for k in [1 .. p] do
        coef1 := 1+(p*k);
        coef2 := (p^2) + 1 - (p*k);
        part1 := chi(gen1, gen2, ch1c1, ch1c2, p, coef1);
        part2 := chi(gen1, gen2, ch2c1, ch2c2, p, coef2);
        gR1[k ][k] := (part1*part2) ;

    end for;
    for l in [1 .. (p^2)] do
        coef := (l + 1) mod (p^2);
        if coef eq 0 then
            coef := p^2;
        end if;
        gR1[p+coef][p+l] := 1;

    end for;
    return gR1;
end function;

action_generator3 := function(gen1, gen2, ch1c1, ch1c2, ch2c1, ch2c2, p)
    dim := p*(p+1);
    gR1 := ZeroMatrix(E,dim,dim);
    for k in [1 .. p] do
        for g in [1 .. p^2] do
            if Valuation(g-gp) ge 2 then
                zgen1 := g;
            end if;
        end for;
        coef1 := inverse(p, zgen1);
        coef2 := (coef1*k) mod p;
        if coef2 eq 0 then
            coef2 := p;
        end if;
        to_mult := chi(gen1, gen2, ch1c1, ch1c2, p, gp);
        gR1[coef2][k]  := to_mult ;

    end for;
    for l in [1 .. (p^2)] do
        coeftemp := (l*gp);
        for c in [1 .. p^2] do
            if Valuation(coeftemp - c) ge 2 then
                coef := c;
            end if;

        end for;
```

```
        gR1[p+coef][p+l] := 1;

    end for;
    return gR1;
end function;

coerc := function(field, M, dim)
    ret := ZeroMatrix(field,dim,dim);
    for i in [1 .. dim] do
        for j in [1 .. dim] do
            ret[i][j] := field!M[i][j];
        end for;
    end for;
    return ret;
end function;


rows_coerc := function(field, M, dim)
    ret := ZeroMatrix(field,dim,dim);
    arr := [];
    for i in [1 .. dim] do
        arr := Append(arr,M[i][1]);

    end for;

    return Matrix(field, dim, dim, arr);

end function;

conv := function(field, x, dim)
    arr:= [];
    for i in [1..dim] do
        arr := Append(arr, x[i]);
    end for;
    return Matrix(field, 1, dim, arr);

end function;


g1:= action_generator1(gen1, gen2, ch1c1, ch1c2, ch2c1, ch2c2, p);
g2:= action_generator2(gen1, gen2, ch1c1, ch1c2, ch2c1, ch2c2, p);
g3:= action_generator3(gen1, gen2, ch1c1, ch1c2, ch2c1, ch2c2, p);

is_unit := function(M)

    if is_int(M) eq 1 then
        Minv:=M^-1;
        if is_int(Minv) eq 1 then
            return 1;
        end if;
```

```
    end if;
    return 0;
end function;

is_stable := function(M)
    if Rank(M) lt d then
        return 0;
    end if;
    Minv := M^-1;
    gM1 := Minv*g1*M;
    if is_int(gM1) eq 1 then
        gM2 := Minv*g2*M;
        if is_int(gM2) eq 1 then
            gM3 := Minv*g3*M;
            if is_int(gM3) eq 1 then
                return 1;

            end if;
        end if;
    end if;
    return 0;
end function;

is_old := function(M, arr) //M rows as in arr
    Minv := M^-1;
    ret:=0;
    for lat in arr do
        ret := ret+1;
        to_check := Minv*lat;
        if is_unit(to_check) eq 1 then
           return ret;

        end if;
    end for;
    return 0;
end function;

scale:= function(M)
    mini:=Valuation(M[1][1]);
    for i in [1.. d] do
        for j in [1..d] do
            mini:= Min(mini, Valuation(M[i][j]));
        end for;
    end for;
    M:= M*(t^(-mini));
    return M;
end function;

find_JH_index:= function(factor, JHarray)
    size := #JHarray;
```

```
    if size gt 0 then
        for ind in [1 .. size] do
            candidate := JHarray[ind];
            if IsIsomorphic(factor, candidate) then
                return ind, JHarray;
            end if;
        end for;
    end if;
    JHarray := Append(JHarray, factor);
    sizenow := size +1;

    return sizenow, JHarray;
end function;

fst :=lattice;
fst:= scale(fst);
all_lat := [fst]; // wrt columns
rec_added := [lattice];
cont := 1;
ll:=1;
graph:=[]; // graph returned will be [[edge1, edge2, dimension_of_quotient, JHfactor]]
JHfactors := [];

while cont gt 0 do
    new_lattices := [];
    for parent in rec_added do
        number_parent := is_old(parent,all_lat);
        R:= parent;// Rbasis to Bbasis
        Rinv:=R^-1;// to Bbasis  Rbasis
        gr1:=Rinv*g1*R;
        gR1:= gr1;
        gr2:=Rinv*g2*R;
        gR2 := gr2;
        gr3:=Rinv*g3*R;
        gR3 := gr3;

        gR1bar := coerc(k_E, gr1, d);
        gR2bar := coerc(k_E, gr2, d);
        gR3bar := coerc(k_E, gr3, d);

        GRbaseLift:=MatrixGroup<d, E| Transpose(gr1),Transpose(gr2), Transpose(gr3)>;
        RbaseLift:=GModule(GRbaseLift);
        GRbaseBar:=MatrixGroup<d, k_E| Transpose(gR1bar),Transpose(gR2bar), Transpose(gR3bar)>;
        Rbase:=GModule(GRbaseBar);

        all_kidds := Submodules(Rbase);

        for kidd in all_kidds do
            quot := quo<Rbase|kidd>;
```

```
if IsIrreducible(quot) then
    Qbasis_L:=[];
    dim_q := Dimension(quot);
    for i in [1 .. dim_q] do
        x:= Rbase!quot.i;
        y := conv(E, x, d);
        y:= t*y;
        Qbasis_L :=Append(Qbasis_L, y);
    end for;
    dim_kidd := Dimension(kidd);
    for i in [1 .. dim_kidd] do
        x:=Rbase!kidd.i;
        y := conv(E, x, d);
        Qbasis_L :=Append(Qbasis_L, y);
    end for;

    Qbasis_L := rows_coerc(E, Qbasis_L, d); //rows
    L:= R* Transpose(Qbasis_L); //wrt Basis, vectors are columns
    L:= scale(L);

    ss := is_stable(L);
    if ss eq 1 then
        nn:=is_old(L,all_lat);

        if nn eq 0 then
            JHindex, JHfactors := find_JH_index(quot, JHfactors);

            ll:=ll+1;
            all_lat:= Append(all_lat, L);
            new_lattices := Append(new_lattices, L);
            graph := Append(graph, [number_parent, ll, dim_q, JHindex]);
        else

            JHindex, JHfactors := find_JH_index(quot, JHfactors);
            graph := Append(graph, [number_parent, nn, dim_q, JHindex]) ;

        end if;

    end if;
end if;
if IsIrreducible(kidd) then
    Qbasis_L:=[];
    dim_q := Dimension(quot);
    for i in [1 .. dim_q] do
        x:= Rbase!quot.i;
        y := conv(E, x, d);
        y:= t*y;
        Qbasis_L :=Append(Qbasis_L, y);
    end for;
    dim_kidd := Dimension(kidd);
```

```
                for i in [1 .. dim_kidd] do
                    x:=Rbase!kidd.i;
                    y := conv(E, x, d);

                    Qbasis_L :=Append(Qbasis_L, y);
                end for;

                Qbasis_L := rows_coerc(E, Qbasis_L, d); //rows
                L:= R* Transpose(Qbasis_L); //wrt Basis, vectors are columns
                L:= scale(L);

                ss := is_stable(L);
                if ss eq 1 then
                    nn:=is_old(L,all_lat);

                    if nn eq 0 then
                        JHindex, JHfactors := find_JH_index(kidd, JHfactors);

                        ll:=ll+1;
                        all_lat:= Append(all_lat, L);
                        new_lattices := Append(new_lattices, L);
                        graph := Append(graph, [ll,number_parent,  dim_kidd, JHindex]);
                    else

                        JHindex, JHfactors := find_JH_index(kidd, JHfactors);
                        graph := Append(graph, [ nn,number_parent, dim_kidd, JHindex]) ;

                    end if;

                end if;
            end if;
        end for;
    end for;
    cont:= #new_lattices;
    rec_added:= new_lattices;
end while;
```

### 4.3.2 Absolutely irreducible representations with abelian $p$-Sylow

Below we show the code used for 2.3.8. As described in more detail in 2.3.8 the code is searching for absolutely irreducible representations with abelian $p$-Sylow subgroup and repeated Jordan Holder factors.

```
order:= 72;
C:=ComplexField();
number_of_groups := NumberOfSmallGroups(order);
divisors := PrimeDivisors(order);
found := [];

for n in [1 .. number_of_groups] do
    G := SmallGroup(order,n);
    T := CharacterTable(G);

    for p in divisors do
        Sylow := SylowSubgroup(G, p);
        if IsAbelian(Sylow) then
            Brauer_p := [];
            I := AbsolutelyIrreducibleModules(G, GF(p));
            for i in I do
                b:= BrauerCharacter(i);
                Brauer_p  := Append(Brauer_p, b);
            end for;

            irregular := [];

            for i in [1 .. #T] do
                null := 0;
                for j in [1 .. #Brauer_p] do
                    if Brauer_p[j][i] eq 0 then
                        null := null + 1;
                    end if;
                end for;
                if null eq (#Brauer_p) then
                    irregular := Append(irregular, i);
                end if;
            end for;

            mtx_bb:= ZeroMatrix(C,#Brauer_p,#Brauer_p);

            for ind_b in [1 .. #Brauer_p] do
                b := Brauer_p[ind_b];
                index := 1;
                for i in [1 .. #b] do
                    if not (i in irregular) then
                        mtx_bb[ind_b][index] := C!b[i];
                        index := index +1;
                    end if;
                end for;
            end for;

            new_t := ZeroMatrix(C,#T,#Brauer_p);

            for ch in [1 .. #T] do
```

```
                t := T[ch];
                index := 1;
                for i in [1 .. #t] do
                    if not (i in irregular) then
                        new_t[ch][index] := C!t[i];
                        index := index +1;
                    end if;
                end for;
            end for;

            decompo := new_t * (mtx_bb^-1);


            for i in [1 .. #T] do
                used := false;
                for j in [1 .. #Brauer_p] do
                    if (Modulus(decompo[i][j]) gt  1.000000000000000000000000001) then
                        found := Append(found, [n, p, i]);
                        used := true;
                    end if;
                end for;
            end for;
        end if;
    end for;
end for;
```

### 4.3.3 Distributivity in a family of lattices

Below we show the code used in the section 2.3. The code is checking if a family of lattices has the distributivity property, defined in 2.1.4, We are iterating over all triples of lattices in a given set of lattices and we check if distributivity is satisfied for all of them. In the implementation of the code we use heavily the Magma's function `HermiteFormBasis()`. We compute the basis of the union of two lattices as the Hermite Normal form of their bases. We compute the basis of the intersection of two lattices as the dual of Hermite Normal form of bases of their duals. For computing the dual basis of a latice we use the fact that if $A$ is the basis matrix of a lattice then the basis matrix of its dual is given by $A(A^TA)^{-1}$.

```
is_int := function(M, d, field)
```

```
    mini := Valuation(M[1][1]);
    for i in [1.. d] do
       for j in [1.. d] do
           mini := Min([mini, Valuation(field!M[i][j])]);
       end for;
    end for;
    if (mini ge 0) then
        return 1;
    end if;
    return 0;
end function;

is_unit := function(M, d, field)
    if not (Valuation(Determinant(M)) eq 0 ) then
        return  0;
    end if;
    if is_int(M, d , field) eq 1 then
        Minv:=M^-1;
        if is_int(Minv, d, field) eq 1 then
            return  1;
        end if;
    end if;
    return 0;
end function;

append_mtxs := function(A, B, dimension)
    M := ZeroMatrix(field, dimension, 2 * dimension);

    for i in [1 .. dimension] do
        for j in [1 .. dimension] do
            M[i, j] := A[i, j];
            M[i, dimension + j] := B[i, j];

        end for;
    end for;
    return M;
end function;

val := function(A, dimension_r, dimension_c)
    //return the minimal valuation of
    //elements in A
    min := Valuation(A[1, 1]);

    for i in [1 .. dimension_r] do
        for j in [1 .. dimension_c] do
            v := Valuation(A[i, j]);
            if (v lt min) then
                min := v;

            end if;
```

```
        end for;
    end for;
    return min;
end function;

convert := function(M, dimension_r, dimension_c, ring_of_integers)
    //convert type of elements
    //in M

    A := ZeroMatrix(ring_of_integers, dimension_r, dimension_c);

    for i in [1 .. dimension_r] do
        for j in [1 .. dimension_c] do
            A[i][j] := (ring_of_integers! M[i][j]);

        end for;
    end for;

    return A;
end function;

HermiteFormBasis := function(M, dimension, ring)
    //extract the basis in Hermite
    //matrix

    A := ZeroMatrix(ring, dimension, dimension);

    for i in [1 .. dimension] do
        for j in [1 .. dimension] do
            A[i][j] := M[i][j];

        end for;
    end for;
    return A;
end function;

union := function(A, B, uniformiser, dimension)
    //we have that the basis of the union is
    // H := HNF([A|B])

    AB := append_mtxs(A, B, dimension);

    v := val(AB, dimension, 2*dimension);

    AB_mult := AB* (uniformiser^(-v));

    AB_mult_ring := convert(AB_mult, dimension, 2*dimension, ring_of_integers);
    H := Transpose(HermiteForm(Transpose(AB_mult_ring)));

    H := convert(H, dimension, 2*dimension, field);
```

```
    ret := HermiteFormBasis(H, dimension, field);
    ret := ret *(uniformiser^(v));

    return ret;

end function;

intersection := function(A, B, uniformiser, dimension)
    //we compute intersection by first computing duals
    // A^v and B^v , then computing H := HNF(A^v | B^v)
    // and the basis matrix of the intersection of A and B
    // is H^v

    // we compute dual of a lattice with basis matrix M
    // as Dual(M) = M(M^{transpose} M)^{-1}

    A_d := A * ((Transpose(A) * A)^-1);
    B_d := B * ((Transpose(B) * B)^-1);

    AB := append_mtxs(A_d, B_d, dimension);

    v := val(AB, dimension, 2*dimension);
    AB_mult := AB* (uniformiser^(-v));

    AB_mult_ring := convert(AB_mult, dimension, 2*dimension, ring_of_integers);
    H := Transpose( HermiteForm( Transpose( AB_mult_ring)));
    H := convert(H, dimension, 2*dimension, field);
    H := HermiteFormBasis(H, dimension, field);

    H := H *(uniformiser^(v));
    H_d := H * ((Transpose(H) * H)^-1);

    return H_d;
end function;

distributive := function(lattices, p, dimension, bound, field, ring_of_integers, uniformiser)
    for i in lattices do
        for j in lattices do
            for k in lattices do
                for ci in [1 .. bound] do
                    for cj in [1 .. bound] do
                        for ck in [1 .. bound] do
                            lat_i := (uniformiser^(ci)) * i;
                            lat_j := (uniformiser^(cj)) * j;
                            lat_k := (uniformiser^(ck)) * k;

                            i_int_k := intersection(lat_i, lat_k, uniformiser, dimension);
                            j_int_k:= intersection(lat_j, lat_k, uniformiser, dimension);
                            i_int_k_un_j_int_k := union(i_int_k, j_int_k, uniformiser, dimension);
```

```
                            i_un_j := union(lat_i, lat_j, uniformiser, dimension);
                            i_un_j_int_k := intersection(i_un_j, lat_k, uniformiser, dimension);

                            to_check := (i_int_k_un_j_int_k^-1) * i_un_j_int_k;
                            eql := is_unit(to_check, dimension, field);

                            if (eql eq 0) then
                                return false;

                            end if;
                        end for;
                    end for;
                end for;
            end for;
        end for;
    end for;

    return true;
end function;
```

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